Open Problems in Representation Theory

Contents

Abstract

In this note, we summarize several open problems that emerged during two events: the Workshop on Harmonic Analysis on Algebraic Groups and the Conference on the Relative Langlands Program, held from August 4 to August 15, 2025, at Tianyuan Mathematics Research Center, Kunming, China. The program was structured in two complementary parts. Each morning featured a minicourse providing an overview of the Relative Langlands Duality proposed by Ben-Zevi, Sakellaridis, and Venkatesh, alongside a series of talks by participants surveying recent advances in this area. In the afternoons, participants divided into focused groups to investigate specific aspects of Relative Langlands Duality, which served as the stimulus for the problems collected here. One of our broader aims is to catalyze attracting more experts, fostering the formulation of more concrete open problems, and thereby shaping future research directions while marking progress in the long-term development of the field.

Acknowledgement

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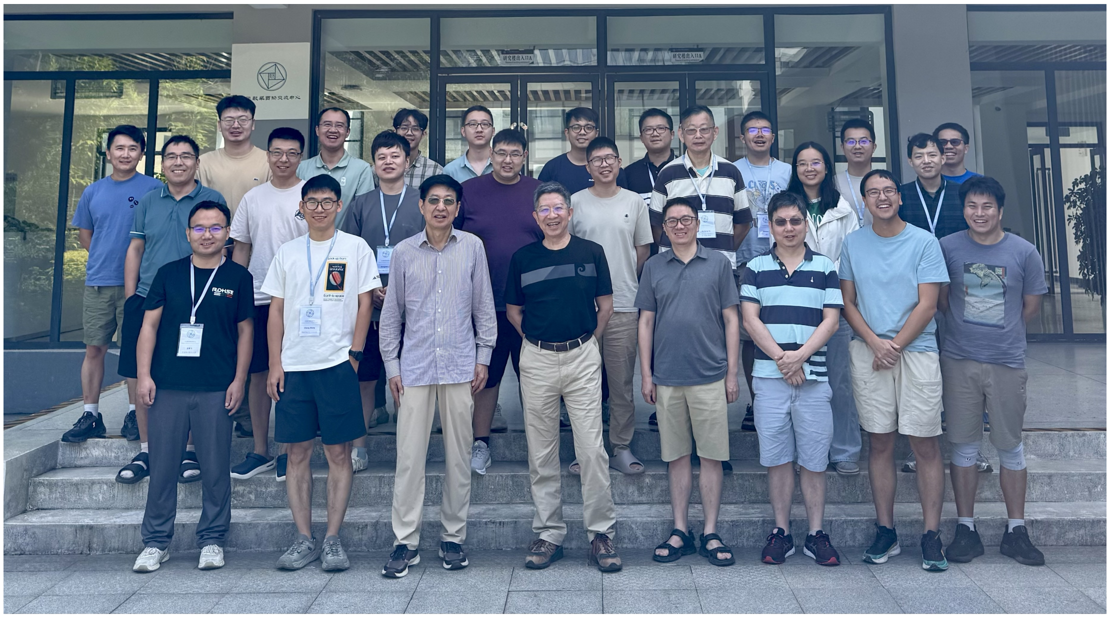

1 Finiteness of local multiplicity [9]

Let F be either a local field or a finite field of sufficiently large characteristic p, and let G be a connected split reductive group defined over F. Suppose M is an anomaly-free hyperspherical variety over F, with associated hyperspherical datum

$$\mathscr{D} = (G, H, S, \iota).$$

Following the notation of [MWZ24, Section 1], one has a unipotent subgroup $U(F)$ together with a Weil representation Θ_ψ of $U(F)$ attached to $\mathscr{D}$. (In the special case where the corresponding nilpotent orbit in $d\iota$ is even, Θ_ψ reduces to the additive character ψ.) We also write Θ_S for the Weil representation associated with the symplectic vector space S.

Over local fields, the following conjecture is natural and widely expected.

Conjecture 1.1. *Let π be an irreducible admissible representation of $G(F)$.*

Then the local multiplicity

$$m_{\mathscr{D}}(\pi) := \dim \operatorname{Hom}_{H(F)\ltimes U(F)} (\pi \otimes \Theta_\psi \otimes \Theta_S, \mathbb{C}) \tag{1}$$

is finite when F is a local field, and independent of the cardinality of F when F is a finite field.

This conjecture may be regarded as a natural extension of the finiteness of multiplicity for spherical varieties G/H, which has been resolved largely through the work of Sakellaridis–Venkatesh, Aizenbud–Gourevitch, and others.

For global applications, the family of hyperspherical varieties satisfying a multiplicity-one property is of particular importance. This motivates the following problem:

Problem 1.1. *Classify all hyperspherical data $\mathscr{D}$ for which*

$$m_{\mathscr{D}}(\pi) \le 1$$

for every irreducible admissible representation π of $G(F)$.

In analogy with the global conjecture of Ben-Zvi–Sakellaridis–Venkatesh (see, for instance, [MWZ24, Conjecture 1.1]), one is further led to ask how a local counterpart of the Relative Langlands duality should be formulated. In particular:

Problem 1.2. *Establish a local multiplicity formula for $m_{\mathscr{D}}(\pi)$ when π is an irreducible admissible* **unitarizable** *representation of $G(F)$.*

2 Identification of twisted cotangent bundle [10]

Let k be a field of characteristic 0. We denote by G_n the reductive group GL_n over k. Let $n \ge 0, m \ge 2$ be integers. For $0 \le r \le m-1$, let N_r be the unipotent subgroup of G_{n+m} of the following shape.

More concretely, it consists of matrices (u_{ij}) with 1 on the diagonal and $u_{ij} \neq 0$ only when $j > i > n$ or $1 \le i \le n, j \ge n+r+1$ or $1 \le j \le n$ and $n+1 \le i \le n+r$.

Let $\lambda_r : N_r \to \mathbb{G}_a$ be the character

$$(u_{ij}) \mapsto \sum_{t=n+1}^{n+m-1} u_{t,t+1}$$

Let $G = G_n \times G_{n+m}$. Let H be the subgroup

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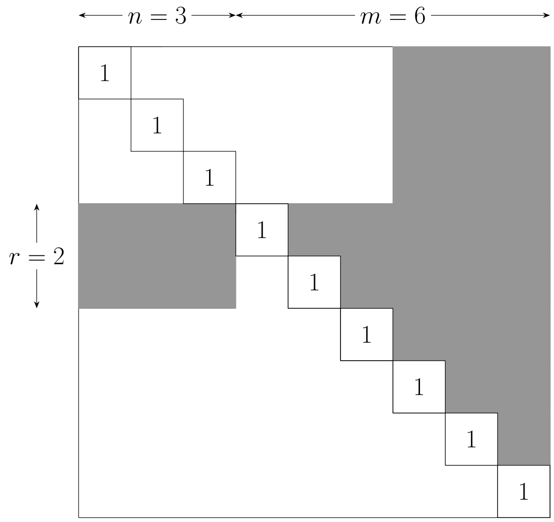

$$\left(h, \begin{pmatrix} h & \\ & I_m \end{pmatrix}\right), \quad h \in G_n$$

of G. By abuse of notation, we also regard N_r as the subgroup $\{1\} \times N_r$ of G.

We introduce the twisted cotangent bundle $T^*(G/N_rH, \lambda_r)$ as

$$T^*(G/N_rH, \psi_r) := (d\lambda_r + (\mathfrak{h} + \mathfrak{n}_r)^\perp) \times^{HN_r} G.$$

Problem 2.1. *The G-Hamiltonian varieties $T^*(G/N_rH, \lambda_r)$ where $0 \le r \le m-1$ are isomorphic to each other.*

Let us explain the motivation for this problem.

We briefly review the local Rankin-Selberg integral.

Let F be a local field. Let N_n denote the upper triangular unipotent subgroup of G_n. Fix an additive character $\psi : F \to \mathbb{C}^\times$. We define a character on $\psi_N : N_n(F) \times N_{n+m}(F)$ by

$$\psi_N(u, u') = \psi\left(\sum_{i=1}^{n-1} u_{i,i+1} - \sum_{j=1}^{n+m-1} u_{j,j+1}\right).$$

Let $\pi = \pi_n \boxtimes \pi_{n+m}$ be an irreducible generic representation of $G(F)$. Denote by $\mathcal{W}(\pi, \psi_N)$ the Whittaker model of π with respect to the character ψ_N. Recall that for any $0 \le r \le m-1$, the local Rankin-Selberg integral Z_r [JPS83] is defined by

$$Z_r(s, W) = \int_{H(F)} \int_{\mathrm{Mat}_{r\times n}(F)} W\left(\begin{pmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{pmatrix} h\right) |\det h|^{s-\frac{m}{2}}\, dx dh.$$

The integral is convergent when $\mathrm{Re}(s) \gg 0$ and has meromorphic continuation to $\mathbb{C}$. The greatest common divisor of $Z_r(W, s)$ for $0 \le r \le m-1$ are the same, and is equal to the local Rankin-Selberg L-function $L(s, \pi_n \times \pi_{n+m})$

Moreover, we have the functional equation

$$\frac{Z_{m-1-r}(1-s, \widetilde{W})}{L(1-s, \pi_n^\vee \times \pi_{n+m}^\vee)} = \omega_{\pi_n}(-1)^{n+m-1} \varepsilon(s, \pi_n \times \pi_{n+m}, \psi) \frac{Z_r(s, W)}{L(s, \pi_n \times \pi_{n+m})},$$

where $\widetilde{W}(g) = W(w_\ell {}^t g^{-1})$, w_ℓ denotes the longest Weyl group element.

Now let $k = F$ be a number field. Let $\mathbb{A} := \mathbb{A}_F$. Fix an additive character $\mathbb{A} \to \mathbb{C}^\times$, let $\psi_r := \psi \circ \lambda_r$. It is a character on $N_r(\mathbb{A})$ trivial on $N_r(F)$.

Let φ be a cusp form on G, we denote by φ_{N_r,ψ_r} the Fourier coefficient of φ along (N_r, ψ_r):

$$\varphi_{N_r,\psi_r}(g) := \int_{[N_r]} \varphi(ug)\psi_r(u)^{-1} du.$$

It is an exercise to show that the zeta integral

$$I_r(s, \varphi) := \int_{[H]} \varphi_{N_r,\psi}(h)|\det h|^{s-\frac{m}{2}} dh$$

decomposes as

$$I_r(s, \varphi) = \prod_v Z_r(s, W_v),$$

whenever we have $W_\varphi = \prod_v W_v$. As a consequence, when $\pi = \pi_n \boxtimes \pi_{n+m}$ is a cuspidal automorphic representation of G, we have $I_r(s, \varphi) \sim L(s, \pi_n \times \pi_{n+m})$. That is, all the integrals I_r give the same L-function.

Following Examples 4.3.11 and 4.3.12 in [BSV24], the period integral I_r is attached to the polarized Hamiltonian variety $T^*(G/HN_r, \lambda_r)$. The problem hence suggests a geometric reason for why there are many ways to give the integral representation of L-function.

3 Socle filtration and intertwining operator [11]

Let F be a local field, $G = G(F)$ a reductive group defined over F, and $P = MN$ a maximal parabolic subgroup of G with M its Levi subgroup. Given σ an irreducible admissible representation of M and $s \in \mathbb{C}$, one forms normalized induced representations $\pi_s := I_P^G(\sigma_s)$ and $\bar{\pi}_s := I_{\bar{P}}^G(\sigma_s)$, where $\bar{P}$ is the unique opposite parabolic subgroup of P.

Definition 3.1. (**level structure**) *For a finite-length admissible representation π of G, we define the level 1 socle $\pi^{(1)}$ of π to be the maximal semi-simple subrepresentation of π, the level 2 socle $\pi^{(2)}$ of π to be the maximal semi-simple subrepresentation of $\pi/\pi^{(1)}$, i.e., there exists a subrepresentation $\pi_{(2)}$ of π such that $\pi_{(2)}/\pi^{(1)} = \pi^{(2)}$, and inductively the level n socle $\pi^{(n)}$ of π to be the maximal semi-simple subrepresentation of $\pi/\pi_{(n-1)}$, i.e., there exists a subrepresentation $\pi_{(n)} \supset \pi_{(n-1)}$ of π such that $\pi_{(n)}/\pi_{(n-1)} = \pi^{(n)}$ for $n \geq 1$, where $\pi_{(0)} := 0$ and only non-zero level structures will be counted by convention. Moreover, we define the level rank $r(\pi)$ of π to be the number of non-zero level structures of π, i.e., the maximal positive integer n such that $\pi/\pi_{(n-1)} \neq 0$.*

Given this, a natural question is

Problem 3.1. *If $\{\pi_s^{(i)} :\ i = 1, 2, \cdots, r\}$ is the socle filtration of π_s, then*

$$\{\bar{\pi}_s^{(i)} := \pi_s^{(r-i+1)} :\ i = 1, 2, \cdots, r\} \text{ is the socle filtration of } \bar{\pi}_s.$$

To state a local "BSD-type" conjecture, we need the notion of standard intertwining operators.

$$M(s) :\ \pi_s \to \bar{\pi}_s, \qquad \bar{M}(s) :\ \bar{\pi}_s \to \pi_s.$$

Assume that $\mathcal{S}(s)$ and $\mathcal{Z}(s)$ (resp. $\bar{\mathcal{S}}(s)$ and $\bar{\mathcal{Z}}(s)$) are the normalization factors of $M(s)$ (resp. $\bar{M}(s)$) such that $M^*(s) := \mathcal{S}(s)M(s)$ (resp. $\bar{M}^*(s) := \bar{\mathcal{S}}(s)\bar{M}(s)$) is holomorphic and non-zero for any $s \in \mathbb{C}$. Moreover, $\bar{M}^*(s) \circ M^*(s) = \mathcal{Z}(s)\bar{\mathcal{Z}}(s)$. In light of Problem 3.1, we propose the following conjecture.

Conjecture 3.1. *Keep the notions as above (see [JL25; Li+25] for more details).*

[11] This section is organized by Caihua Luo (luocaihua@cuhk.edu.cn)

(i) (local "BSD-type" formula)

$$r(\pi_s) = \mathrm{ord}_s(\mathcal{Z}(s)\bar{\mathcal{Z}}(s)) + 1.$$

(ii) ("no gap" property)

$$\mathrm{ord}_s(M^*(s)|_{\pi_s^{(i+1)}}) = \mathrm{ord}_s(M^*(s)|_{\pi_s^{(i)}}) - 1 \text{ for any } 1 \le i \le r(\pi_s) - 1,$$

and for any $1 \le i \le r(\pi_s)$ and any subrepresentations $\pi_1, \pi_2 \subset \pi_s^{(i)}$,

$$\mathrm{ord}_s(M^*(s)|_{\pi_1}) = \mathrm{ord}_s(M^*(s)|_{\pi_2}).$$

Next, let us examine an interesting example that follows. Let $G_n = GL_n(F)$ be the general linear group of rank n defined over a p-adic field F. Given a unitary supercuspidal representation τ of G_m and two positive integers c, a, we can attach a Speh representation $\rho_c(\tau_a)$, then form the induced representations of G_n

$$\pi_s := \rho_c(\tau_a)|\det(\cdot)|^s \times \rho_d(\tau_b)|\det(\cdot)|^{-s} \text{ and } \bar{\pi}_s := \rho_d(\tau_b)|\det(\cdot)|^{-s} \times \rho_c(\tau_a)|\det(\cdot)|^s,$$

thus the corresponding standard intertwining operator

$$M(s) : \pi_s \to \bar{\pi}_s.$$

Conjecturally, we know that the normalization factors $\mathcal{S}(s)$ and $\mathcal{Z}(s)$ are given by

$$\mathcal{S}(s)^{-1} = \prod_{j=\frac{|c-d|}{2}}^{\frac{c+d-2}{2}} L(2s - j, \tau_a \times \tau_b^\vee), \qquad \mathcal{Z}(s)^{-1} = \prod_{j=\frac{|c-d|}{2}}^{\frac{c+d-2}{2}} L(2s + j + 1, \tau_a \times \tau_b^\vee).$$

That is to say,

Conjecture 3.2. *(see [Luo25] for the detail) Use the same notation as above. We have*

$$M^*(s) := \mathcal{S}(s)M(s)$$

is always non-zero for $s \in \mathbb{C}$, i.e., a non-zero intertwining operator.

In view of the above Conjecture 3.2 and Tadić's expectation in [Tad15], our Conjecture 3.1 predicts the following conjecture.

Conjecture 3.3. *Each level of the socle filtration of π_s is irreducible, i.e., π_s is a uniserial module.*

Remark 3.1. *The non-zero conjecture of $M^*(s)$ is shown for the case $c = d$ in [Luo25], and the uniserial property for the case $(c = d, a = b)$ is known in [Tad15].*

4 Sheaves on buildings and representations of p-adic reductive groups [12]

Let G be a p-adic reductive group. For a fixed non-negative real number r, the Moy-Prasad filtration defines a map

$$\begin{aligned} B(G) &\longrightarrow \{\text{compact subgroup of } G\} \\ x &\longmapsto G_{x,r^+} \end{aligned}$$

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This construction gives rise to a cellular structure of $B(G)$ where x and y are in the same r-facet if and only if $G_{x,r^+} = G_{y,r^+}$. For each facet σ, let $G_\sigma := G_{x,r^+}$ where x is any point in the interior of σ. Let $C_d(B(G))$ denote the free abelian group generated by d-dimensional r-facets of $B(G)$.

For a complex G-representation V and a subgroup $K \leq G$, let V^K denote the subspace of K-fixed vectors. Following Schneider-Stuhler's construction [SS97], Bestvina and Savin [BS20] defined a complex $(C_*(B(G), V), \partial)$ with

$$C_*(B(G), V) := \mathrm{span}\{\sigma \otimes v | v \in V^{G_\sigma}\} \subset C_*(B(G)) \otimes_{\mathbb{Z}} V.$$

Suppose V is a G-module generated by $\bigcup_{x\in B(G)} V^{x,r^+}$. Bestvina and Savin showed that $(C_*(B(G), V), \partial)$ gives a projective resolution of V. For example, consider $r = 0$ and the subcategory $\mathrm{Rep}_0(G)$ (which is a direct summand of the category of smooth G-module) generated by depth-zero representations of G. Then $C_*(B(G), V)$ is a projective resolution for every $V \in \mathrm{Rep}_0(G)$.

We suggest exploring the possibility of using refined K-types following the work of Kim-Yu and Adler-Fintzen-Mishra-Ohara. We retain some notation from [Adl+24, §3.6]. Let $\Sigma := (\overrightarrow{G}, \overrightarrow{r}, \overrightarrow{\phi})$ be a (partial) Heisenberg-Weil datum [13] Now the Kim-Yu-AFMO construction defines a map from $B(G^0)$ to quadruple

$$B(G^{(0)}) \ni x \mapsto (K_x, K_{x,+}, \phi_x, \kappa_x)$$

where $K_{x,+}$ is a finite index normal subgroup of the open compact subgroup K_x and ϕ_x is a character of $K_{x,+}$, κ_x is the twisted Yu Heisenberg-Weil representation of K_x (an irreducible K_x-representation such that $\kappa_x|_{K_{x,+}}$ is ϕ_x-isotypic and satisfies some compatibility condition for different x). Similar to the definition of the building, we define

$$B_\Sigma := G \times A^{(0)} / \sim$$

where

(i) $A^{(0)}$ is the apartment attached to a fixed maximal split torus $T^{(0)}$ of $G^{(0)}$

(ii) $(g, x) \sim (h, y)$ if and only if there is $n \in N(T^{(0)})$ such that $n \cdot x = y$ and $g^{-1}hn \in K_x$.

(iii) the cellular structure on $A^{(0)}$ is defined as follows: x and y are in the same facet if $K_x = K_y$.

It is known from Kim-Yu [KY17] that the subcategory $\mathrm{Rep}_\Sigma(G)$ of G-modules generated by $\bigcup_{x\in B(G^0)} V[\phi_x]$ is a direct summand of the category of smooth G-modules. For any $V \in \mathrm{Rep}_\Sigma(G)$, we define

$$C_*(B_\Sigma, V) := \mathrm{span}\{\sigma \otimes v | v \in V[\phi_x]\} \subseteq C_*(B_\Sigma) \otimes_{\mathbb{Z}} V$$

where $V[\phi_x]$ denotes the ϕ_x-isotypic component of V.

Conjecture 4.1. *The complex $C_*(B_\Sigma, V)$ is a projective resolution of V when $V \in \mathrm{Rep}_\Sigma(G)$.*

Following Schneider-Stuhler, one can consider G-equivariant coefficient systems $\mathscr{V}$ on B_Σ by requiring a space $\mathscr{V}_\sigma$ with K_x-action attached to each simplex σ such that $\mathscr{V}_\sigma$ is ϕ_x-isotypic. Let $\mathrm{Coeff}_G(B_\Sigma, \Sigma)$ denote the category of such kinds of coefficient systems. Observe that $B(G^0) := G^{(0)} \times A^{(0)} / \sim$ is naturally embedded into $B_\Sigma := G \times A^{(0)} / \sim$. There is a natural restriction map

$$\begin{array}{ccc} \mathrm{Coeff}_G(B_\Sigma, \Sigma) & \longrightarrow & \mathrm{Coeff}_{G^{(0)},0}(B(G^{(0)})) \\ \mathscr{V} & \longmapsto & \mathrm{Res}\mathscr{V} \end{array}$$

[13] This datum defines an "endo-class" in the language of Bushnell-Kutzko-Stevens for classical groups.

sending $\mathscr{V}$ to $(\mathrm{Res}\mathscr{V})_\sigma := \mathrm{Hom}_{(K_x\cap G_{x,0^+})}(\kappa_x, \mathscr{V}_\sigma)$. Here $\mathrm{Coeff}_{G^{(0)},0}(B(G^{(0)}))$ denotes the $G^{(0)}$-equivariant coefficient system on $B(G^{(0)})$ such that a space V_σ with $G^{(0)}_x$-action is attached to each facet σ and $G^{(0)}_{x,0^+}$ acts trivially ($x \in \sigma$).

One can define a homology functor H_0 from $\mathrm{Coeff}_G(B_\Sigma)$ to $\mathrm{Rep}_\Sigma(G)$ by the same recipe in [SS97, §V].

Conjecture 4.2 (Compare with [SS97, Theorem V.1])**.** *The following diagram induces an equivalence of categories between* $\mathrm{Rep}_\Sigma(G)$ *and* $\mathrm{Rep}_0(G^{(0)})$

$$\mathrm{Rep}_\Sigma(G) \xleftarrow{H_0} \mathrm{Coeff}_G(B_\Sigma, \Sigma)[H_0^{-1}] \xrightarrow{\mathrm{Res}} \mathrm{Coeff}_{G^{(0)},0}(B(G^{(0)}))[H_0^{-1}] \xrightarrow{H_0} \mathrm{Rep}_0(G^{(0)}).$$

Here "$[H_0^{-1}]$*" denotes the localization with respect to morphisms* s *that induce isomorphisms* $H_0(s)$*.*

5 Archimedean type II theta correspondence, coherent continuations and Whittaker models [14]

Let G be a finite central extension of an open subgroup of $\mathbb{R}$-points of a connected reductive algebraic group (reductive Nash group [Sun15]) with Lie algebra $\mathfrak{g}$. Fix an invariant nondegenerate bilinear form $B(-,-)$ on $\mathfrak{g}$.

For an irreducible Casselman-Wallach representation π of G, let $\mathrm{Wh}_{\mathcal{O}}(\pi)$ be the space of generalized Whittaker models of π associated to the nilpotent orbit $\mathcal{O}$. Let

$$\mathrm{WO}(\pi) := \{\mathcal{O} \,|\, \mathcal{O} \text{ is a nilpotent orbit such that } \mathrm{Wh}_{\mathcal{O}}(\pi) \neq 0\},$$

and the *Whittaker support* $\mathrm{WS}(\pi)$ of π be the set of all maximal orbits in $\mathrm{WO}(\pi)$ with respect to closure inclusion:

$$\mathrm{WS}(\pi) := \{\mathcal{O} \,|\, \mathcal{O} \text{ is a maximal orbit in } \mathrm{WO}(\pi)\}.$$

Another nilpotent invariant is the wavefront cycle:

$$\mathcal{WF}(\pi) = \sum_i m_i[\mathcal{O}_i].$$

With the corresponding wavefront set :

$$\mathrm{WF}(\pi) = \bigcup_{m_i\neq 0} \bar{\mathcal{O}}_i.$$

Mœglin-Waldspurger [MW87] proved that $\mathrm{WF}(\pi) = \mathrm{WS}(\pi)$ when F is a p-adic field. We want to test this conjecture for arbitrary reductive almost linear Nash groups.

Conjecture 5.1 (Main conjecture)**.** *For any reductive almost linear Nash group* G *and any irreducible Casselman-Wallach representation* π*:*

$$\mathrm{WF}(\pi) = \mathrm{WS}(\pi).$$

Take a nilpotent orbit $\mathcal{O} \subseteq \mathfrak{g}$ (or $\mathfrak{sl}_2$-triple $\gamma = \{Y, H, X\}$ in $\mathfrak{g}$). $\mathfrak{g}/\mathfrak{g}^Y$ is a symplectic space with symplectic form $\langle -,-\rangle := B(Y, [-,-])$. Set $G^Y := \mathrm{Stab}_G(Y)$, there is a natural homomorphism $G^Y \to \mathrm{Sp}(\mathfrak{g}/\mathfrak{g}^Y, \langle -,-\rangle)$. Let $\widetilde{G^Y}$ be the double cover of G^Y induced by the metaplectic double cover of $\mathrm{Sp}(\mathfrak{g}/\mathfrak{g}^Y, \langle -,-\rangle)$.

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Definition 5.1. *The nilpotent $\mathcal{O}$ is called admissible if the double cover $\widetilde{G^Y}$ is split over the identity component of $(G^Y)_0$. It is called quasi-admissible if $\widetilde{G^Y}$ has a finite-dimensional genuine representation.*

Result of [GGS21] says that $\mathrm{WS}(\pi)$ consists of quasi-admissible orbits. But they do not know whether the non-admissible quasi-admissible orbits appear in Whittaker supports of representations. We may ask the same question for the wavefront set:

Problem 1: Are all the nilpotent orbits in the wavefront cycle admissible?

For the case when G is general linear, it is well known that the wavefront set $\mathrm{WF}(\pi)$ of an irreducible Casselman-Wallach representation π of G consists of a single nilpotent adjoint orbit. Together with some counting results, it may be possible to construct all irreducible representations of the general linear group via theta correspondence and coherent continuation.

Problem 2: Can we construct all irreducible representations via theta correspondence, coherent continuation, and twist of characters?

Assuming the construction steps above are done, it then remains to study the Whittaker support $\mathrm{WS}(\pi)$ of π. We focus on two types of interactions with Whittaker models - theta correspondence and coherent continuation.

On one hand, Gomez-Zhu [GZ14] proved for type I reductive (symplectic-orthogonal) dual pairs in the stable range that the big theta lift preserves the structure of Whittaker models:

$$\mathrm{Wh}_{\Theta(\mathcal{O})}(\Theta(\pi)) = \mathrm{Wh}_{\mathcal{O}}(\pi^\vee).$$

Noting that $(G, G') = (\mathrm{GL}_n(F), \mathrm{GL}_m(F))$ is a (reductive) dual pair for any $n, m \in \mathbb{N}$, one observes that if one can prove an analogue of Gomez-Zhu's result for type II dual pairs, one can use it to infer properties of the structure of the Whittaker support $\mathrm{WS}(\pi)$. Hence, we propose the following subconjecture:

Conjecture 5.2. *If $(G, G') = (\mathrm{GL}_n(F), \mathrm{GL}_m(F))$ is a type II reductive dual pair over F (with $n \le m$), and π is a genuine Casselman-Wallach representation of G, then*

$$\mathrm{Wh}_{\Theta(\mathcal{O})}(\Theta(\pi)) = \mathrm{Wh}_{\mathcal{O}}(\pi^\vee),$$

where $\pi^\vee$ denotes the smooth contragredient of π. Furthermore, the above will imply that

$$\mathrm{WS}(\Theta(\pi)) = \mathrm{WS}(\pi^\vee).$$

On the other hand, Barbasch-Vogan [BV80] proved that under coherent continuation, the multiplicity of wavefront cycle changes in the form of a Goldie rank polynomial, i.e.

$$\mathcal{WF}(\pi(\lambda)) = p(\lambda) \cdot (\sum_i c_i[\mathcal{O}_i]),$$

where $p(\lambda)$ is the goldie rank polynomial corresponding to the coherent family $\pi(\mu)$, λ is dominant. We expect some similar results for the multiplicity of the generalized Whittaker model.

Problem 3: For a fixed nilpotent orbit $\mathcal{O}$, does there exist a constant $c_{\mathcal{O}}$ such that $\dim \mathrm{Wh}_{\mathcal{O}}(\pi(\lambda)) = c_{\mathcal{O}} \cdot p(\lambda)$?

6 Asymptotic of Sobolev norms [15]

Let M be a (nonempty) complete Riemannian manifold. The Laplacian-Beltrami operator on M is denoted by Δ. Let $C_c^\infty(M)$ denote the space of complex-valued smooth functions on M with compact support. For each real number s, define an inner product on $C_c^\infty(M)$ by

$$\langle f, g\rangle_s = \int_M ((1-\Delta)^s.f)(x) \cdot \overline{g(x)}\, dx,$$

[15] Question posed by Binyong Sun. This section is organized by Dongwen Liu (maliu@zju.edu.cn)

where dx is the measure on M induced by the Riemannian metric.

Let N be a closed submanifold of M and let μ be a smooth measure on N (namely under the local coordinates it is the product of a smooth function with the Lebesgue measure). Consider the functional

$$\lambda_\mu : C_c^\infty(M) \to \mathbb{C}, \quad f \mapsto \int_N f(x)d\mu(x).$$

Define the Sobolev norm

$$|\mu|_s := \sup_{f\in C_c^\infty(M), \langle f,f\rangle_s \leq 1} |\lambda_\mu(f)|.$$

Problems: 1. Find the conditions on (M, N, μ, s) that $|\mu|_s < \infty$.

2. Describe asympototic of $|\mu|_s$ when $s \to \infty$.

One interesting case is when M is a modular curve viewed as a Riemannian manifold.

7 Some local theory of unitary Friedberg-Jacquet periods [16]

Let F be a local field and E/F be a quadratic field extension. Take a non-degenerate Hermitian space $(W, \langle -, -\rangle)$ over E of dimension $2n$. Let $G = U(W)$ be unitary group and $\tilde{G} = \mathrm{GU}(W)$ be the unitary similitudes group.

The first problem concerns the Adler-Prasad conjecture on the restriction of irreducible $\tilde{G}$-representations to G. Let G^L and $\tilde{G}^L$ be the Langlands dual group respectively. The injection $\iota : G \hookrightarrow \tilde{G}$ induces a morphism $\iota^L : \tilde{G}^L \to G^L$. As a special case of [AP19], we have the following conjecture:

Conjecture 7.1 (Adler-Prasad)**.** *Assume the local Langlands correspondence for G and $\tilde{G}$. Let $\tilde{\pi}$ (resp. π) be irreducible admissible representation of $\tilde{G}$ (resp. G) with corresponding Langlands parameter $\tilde{\phi}$ (resp. ϕ). Then $\mathrm{Hom}_{G(F)}(\tilde{\pi}, \pi) = 0$ unless $\phi = \iota^L \circ \tilde{\phi}$. In this case,*

$$\dim \mathrm{Hom}_{G(F)}(\tilde{\pi}, \pi) = \dim \mathrm{Hom}_{\pi_0(Z(\tilde{\phi}))}(\tilde{\eta}, \iota^{L,*}\eta)$$

Here $\tilde{\eta}$ (resp. η) is the character of component group of $\tilde{\phi}$ (resp. ϕ) corresponding to $\tilde{\pi}$ (resp. π).

We refer to [CZ21] for the local Langlands correspondence for G and to [KKS25] for the generic local Langlands correspondence for quasi-split $\tilde{G}$. We expect that one can apply the theta lifting method in [CZ21] to complete the local Langlands correspondence for $\tilde{G}$ general. Our motivation for this conjecture is the potential role of $\tilde{G}$ in the special value formula of unitary Friedberg-Jacquet period.

The second problem concerns the Plancherel measure of $L^2(H(F)\backslash G(F), \omega)$. Here $H \subset G$ is a suitable subgroup and $\omega : H(F) \to \mathbb{C}^\times$ is a unitary character. We shall take H to be

- $U(V) \times U(V^\perp)$ when $W = V \oplus V^\perp$ and $\dim V = n$;
- $\mathrm{GL}(X)$ when $W = X \oplus Y$ is maximally split, i.e. X and Y are totally isotropic of dimension n.

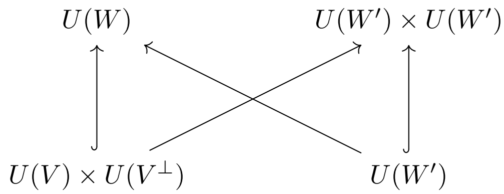
Assume $\omega = 1$ and W is maximally split. By [CG22], for any irreducible tempered G-representation, there exists a uniquely determined (skew)-Hermitian space W' of dimension $2n$ such that

$$\dim \mathrm{Hom}_{\mathrm{GL}(X)}(\pi, \mathbb{C}) = \dim \sum_V \mathrm{Hom}_{U(V)\times U(V^\perp)}(\Theta(\pi), \mathbb{C})$$

Here the $U(W')$-representation $\Theta(\pi)$ is the theta lifting of π and $V \subset W'$ runs the isomorphism classes of non-degenerate (skew)-Hermitian subspace of dimension n. In [CG22], the authors

[16] This section is organized by Yangyu Fan (yangyu.fan@bit.edu.cn)

use the unitary Shalika period as a bridge to establish the identity. We expect such identity can deduced directly from the seesaw diagram



and the local Siegel-Weil formula for theta lifting from $U(V)$ to $U(W)$. Bearing the work [GG14] in mind, we wonder if this approach can allow one to deduce the Plancherel measure of $L^2(\mathrm{GL}(X)\backslash U(W))$ from that of $L^2(U(V)\times U(V^{\perp})\backslash U(W'))$ for all possible $V \subset W'$.

Finally, could one deduce the Plancherel measure of $L^2(H(F)\backslash G(F), \omega)$ for general ω from that of $L^2(H(F)\backslash G(F))$? This question is motivated by further arithmetic applications in Euler systems and p-adic deformation of periods.

8 Invariant tempered distributions: composition structure[17]

Consider the reductive dual pair

$$(H, G) := (\mathrm{O}_{p,q}, \mathrm{Sp}_{2n}(\mathbb{R})) \subseteq \mathrm{Sp}_{2N}(\mathbb{R}), \quad N = (p+q)n,$$

and the smooth oscillator representation ω of the metaplectic group $\widetilde{\mathrm{Sp}}_{2N}(\mathbb{R})$ on $\mathcal{S}(\mathrm{M}_{p+q,n}(\mathbb{R}))$, the space of Schwartz class functions in $\mathrm{M}_{p+q,n}(\mathbb{R})$. As usual, we normalize ω so that the action of $\widetilde{H}$ factors through the obvious action of H on $\mathcal{S}(\mathrm{M}_{p+q,n}(\mathbb{R}))$. We have the dualized action of $\widetilde{\mathrm{Sp}}_{2N}(\mathbb{R})$ on $\mathcal{S}^*(\mathrm{M}_{p+q,n}(\mathbb{R}))$, the space of tempered distributions on $\mathrm{M}_{p+q,n}(\mathbb{R})$, and it is denoted by ω^*.

Let

$$\begin{aligned}\Omega^{p,q}(\mathbb{1}) = &\text{Howe's maximal quotient corresponding to} \\ &\text{the trivial representation } \mathbb{1} \text{ of } \mathrm{O}_{p,q}.\end{aligned}$$

The continuous dual of $\Omega^{p,q}(\mathbb{1})$ is

$$\mathcal{I} := \mathcal{S}^*(\mathrm{M}_{p+q,n}(\mathbb{R}))^{\mathrm{O}_{p,q}},$$

the space of $\mathrm{O}_{p,q}$-invariant tempered distributions on $\mathrm{M}_{p+q,n}(\mathbb{R})$.

Basic facts: (Kudla-Rallis [KR90])

(a).

$$\mathcal{I} = \overline{< \omega^*(\widetilde{G})\delta >},$$

where δ is the Dirac measure at the origin of $\mathrm{M}_{p+q,n}(\mathbb{R})$, and $\overline{< D >}$ denotes closure of the span of a set D in the standard Frechet topology of $\mathcal{S}^*(\mathrm{M}_{p+q,n}(\mathbb{R}))$.

(b). There is a natural embedding

$$\Omega^{p,q}(\mathbb{1}) \hookrightarrow \mathrm{Ind}_{\widetilde{M}N}^{\widetilde{G}}(\chi_0^{p-q} \otimes |\;|^{\frac{p+q-(n+1)}{2}} \otimes 1),$$

where $M \simeq \mathrm{GL}_n(\mathbb{R})$, and χ_0 is the character of $\widetilde{M}$ of order 4 given by

$$\chi_0(a, \epsilon) = \epsilon \cdot \begin{cases} 1, & \text{if } \det(a) > 0, \\ i, & \text{if } \det(a) < 0. \end{cases}$$

[17]This section is organized by Chen-Bo Zhu (matzhucb@nus.edu.sg)

(The embedding is induced by the map

$$\psi_{p,q}(f)(\tilde{g}) = \delta(\omega(\tilde{g})f), \qquad f \in \mathcal{S}(\mathrm{M}_{p+q,n}(\mathbb{R})),\ \tilde{g} \in \widetilde{G}.)$$

From the work of Lee-Zhu [LZ97], we know the composition structure of $\Omega^{p,q}(\mathbf{1})$ and their relationship with the structure of $\mathrm{Ind}_{\widetilde{MN}}^{\widetilde{G}}(\chi \otimes |\ |^s \otimes 1)$. The results are described in terms of $\widetilde{K}$-structures. Here $K \simeq \mathrm{U}_n$ is a maximal compact subgroup of G.

Problem 1: Describe the composition structure of $\mathcal{I}$ in terms of the geometry of H-action on $\mathrm{M}_{p+q,n}(\mathbb{R})$.

According to the Howe Duality Theorem [How89], $\Omega^{p,q}(\mathbf{1})$ has a unique irreducibe $\widetilde{G}$-quotient, denoted by $\theta^{p,q}(\mathbf{1})$. Dually it says that $\mathcal{I}$ has a unique irreducible $\widetilde{G}$-submodule, denoted by $\mathcal{J}$.

Problem 2: Determine $\mathcal{J}$, by finding a non-zero element of $\mathcal{J}$.

Remark: If $\Omega^{p,q}(\mathbf{1})$ is irreducible, for example, this is the case when $pq = 0$ (compact dual pair) or $p + q \leq n$ (stable range), then of course $\mathcal{J} = \mathcal{I}$, and so one can pick any non-zero element of $\mathcal{I}$ (such as δ). Problem 2 is trivial in this case.

Example: Consider the null cone:

$$\mathcal{N} = \{v \in \mathrm{M}_{p+q,n}(\mathbb{R}) |\, v^t I_{p,q} v = 0 \quad \text{(the } n \times n \text{ zero matrix)}\},$$

where $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. Clearly $\mathcal{N}$ is stable under H.

Suppose $p, q \geq n$. The regular part of the null cone defined by

$$\mathcal{N}_n = \{v \in \mathcal{N} |\, \mathrm{rank}(v) = n\}$$

consists of a single H-orbit, and carries an H-invariant Radon measure, denoted by $d\mu_n$. In fact $d\mu_n$ defines a tempered distribution on $\mathrm{M}_{p+q,n}(\mathbb{R})$.

Set

$$\omega^*(d\mu_n) = \overline{< \omega^*(\widetilde{G}) d\mu_n >}.$$

Then the following results hold: ([HZ02])

(a). $\omega^*(d\mu_n)$ is the unique irreducible $\widetilde{G}$-submodule of $\mathcal{I}$.

(b). $\omega^*(d\mu_n)$ is finite dimensional if and only if $p, q \equiv n + 1 \pmod 2$.

References

[BV80] Dan Barbasch and David A. Vogan Jr. “The local structure of characters”. In: *J. Functional Analysis* 37.1 (1980), pp. 27–55. ISSN: 0022-1236. DOI: 10.1016/0022-1236(80)90026-9. URL: https://doi.org/10.1016/0022-1236(80)90026-9.

[JPS83] Hervé Jacquet, Ilya I Piatetskii-Shapiro, and Joseph A Shalika. “Rankin-selberg convolutions”. In: *American journal of mathematics* 105.2 (1983), pp. 367–464.

[MW87] C Mœglin and J L Waldspurger. “Modèles de Whittaker dégénérés pour des groupes p-adiques”. In: *Mathematische Zeitschrift* 196 (3 1987), pp. 427–452. ISSN: 1432-1823. DOI: 10.1007/BF01200363. URL: https://doi.org/10.1007/BF01200363.

[How89] R. Howe. “Transcending classical invariant theory”. In: *J. Amer. Math. Soc.* 2 (1989), pp. 535–552.

[KR90] S.S. Kudla and S. Rallis. “Degenerate principal series and invariant distributions”. In: *Israel J. Math.* 69 (1990), pp. 25–45.

[LZ97] S.T. Lee and C.-B. Zhu. “Degenerate principal series and local theta correspondence II”. In: *Israel J. Math.* 100 (1997), pp. 29–59.

[SS97] Peter Schneider and Ulrich Stuhler. “Representation theory and sheaves on the Bruhat-Tits building”. en. In: *Publications Mathématiques de l'IHÉS* 85 (1997), pp. 97–191. URL: https://www.numdam.org/item/PMIHES_1997__85__97_0/.

[HZ02] R. Howe and C.-B. Zhu. “Eigendistributions for orthogonal groups and representations of symplectic groups”. In: *J. Reine Angew. Math.* 545 (2002), pp. 121–166.

[GG14] Wee Teck Gan and Raul Gomez. “A conjecture of Sakellaridis-Venkatesh on the unitary spectrum of spherical varieties”. In: *Symmetry: representation theory and its applications*. Vol. 257. Progr. Math. Birkhäuser/Springer, New York, 2014, pp. 185–226. ISBN: 978-1-4939-1589-7; 978-1-4939-1590-3. DOI: 10.1007/978-1-4939-1590-3_7. URL: https://doi.org/10.1007/978-1-4939-1590-3_7.

[GZ14] Raul Gomez and Chen-Bo Zhu. “Local theta lifting of generalized Whittaker models associated to nilpotent orbits”. In: *Geometric and Functional Analysis* 24 (3 2014), pp. 796–853. ISSN: 1420-8970. DOI: 10.1007/s00039-014-0276-5. URL: https://doi.org/10.1007/s00039-014-0276-5.

[Sun15] Binyong Sun. “Almost linear Nash groups”. In: *Chinese Ann. Math. Ser. B* 36.3 (2015), pp. 355–400. ISSN: 0252-9599,1860-6261. DOI: 10.1007/s11401-015-0915-7. URL: https://doi.org/10.1007/s11401-015-0915-7.

[Tad15] Marko Tadić. “On the reducibility points beyond the ends of complementary series of p-adic general linear groups”. In: *J. Lie Theory* 25.1 (2015), pp. 147–183. ISSN: 0949-5932.

[KY17] Ju-Lee Kim and Jiu-Kang Yu. “Construction of Tame Types”. In: *Representation Theory, Number Theory, and Invariant Theory: In Honor of Roger Howe on the Occasion of His 70th Birthday*. Ed. by Jim Cogdell, Ju-Lee Kim, and Chen-Bo Zhu. Cham: Springer International Publishing, 2017, pp. 337–357. ISBN: 978-3-319-59728-7. DOI: 10.1007/978-3-319-59728-7_12. URL: https://doi.org/10.1007/978-3-319-59728-7_12.

[AP19] Jeffrey D. Adler and Dipendra Prasad. “Multiplicity upon restriction to the derived subgroup”. In: *Pacific J. Math.* 301.1 (2019), pp. 1–14. ISSN: 0030-8730,1945-5844. DOI: 10.2140/pjm.2019.301.1. URL: https://doi.org/10.2140/pjm.2019.301.1.

[BS20] Mladen Bestvina and Gordan Savin. “Bounded contractions for affine buildings”. In: *Proceedings of the American Mathematical Society* 148 (2020). MR4052222, pp. 875–883. DOI: 10.1090/proc/14877.

[CZ21] Rui Chen and Jialiang Zou. “Local Langlands correspondence for unitary groups via theta lifts”. In: *Represent. Theory* 25 (2021), pp. 861–896. ISSN: 1088-4165. DOI: 10.1090/ert/588. URL: https://doi.org/10.1090/ert/588.

[GGS21] Raul Gomez, Dmitry Gourevitch, and Siddhartha Sahi. “Whittaker supports for representations of reductive groups”. en. In: *Annales de l'Institut Fourier* 71.1 (2021), pp. 239–286. DOI: 10.5802/aif.3372. URL: https://aif.centre-mersenne.org/articles/10.5802/aif.3372/.

[CG22] Rui Chen and Wee Teck Gan. *Unitary Friedberg-Jacquet periods*. 2022. arXiv: 2108.04064 [math.RT]. URL: https://arxiv.org/abs/2108.04064.

[Adl+24] Jeffrey D. Adler et al. “Reduction to depth zero for tame p-adic groups via Hecke algebra isomorphisms”. In: (2024). arXiv: 2408.07805 [math.RT]. URL: https://arxiv.org/abs/2408.07805.

[BSV24] David Ben-Zvi, Yiannis Sakellaridis, and Akshay Venkatesh. *Relative Langlands Duality*. 2024. arXiv: 2409.04677 [math.RT]. URL: https://arxiv.org/abs/2409.04677.

[MWZ24] Zhengyu Mao, Chen Wan, and Lei Zhang. *Strongly tempered hyperspherical Hamiltonian spaces*. 2024. arXiv: 2405.17699 [math.NT]. URL: https://arxiv.org/abs/2405.17699.

[JL25] Chris Jantzen and Caihua Luo. "DEGENERATE PRINCIPAL SERIES FOR BRYLINSKI-DELIGNE COVERS OF SPLIT GROUPS OF CLASSICAL TYPE". In: *Preprint on researchgate RG.2.2.30792.43521* (2025).

[KKS25] Yeansu Kim, Muthu Krishnamurthy, and Freydoon Shahidi. *L-packets and the generic Arthur packet conjectures for even unitary similitude groups*. 2025. arXiv: 2506.00892 [math.NT]. URL: https://arxiv.org/abs/2506.00892.

[Li+25] Ning Li et al. "DEGENERATE PRINCIPAL SERIES FOR REAL GROUPS OF CLASSICAL TYPE". In: *Preprint on researchgate RG.2.2.12633.63840* (2025).

[Luo25] Caihua Luo. "Singularity of intertwining operator and reducibility of parabolic induction for $GL_n(F)$". In: *submitted* (2025).