Open Problems in Domain Theory

on the occasion of the Domain Theory Seminar @ TYMC, December 2025.

Contents

A conjecture on directed spaces

Hui Kou
School of Mathematics, Sichuan University,
Chengdu, Sichuan 610064, China
kouhui@scu.edu.cn

Conjecture: **DTOP** is the largest cartesian closed full subcategory of $\mathbf{TOP_0}$ such that containing all c-spaces and all posets with the Scott topologies.

The theory of domains, which arose from logic and computer science and started with the works of Dana Scott[1, 2, 3, 4], is a foundation of the semantics of programming languages, logic and lambda calculus. After about 40 years of domain theory, one is forced to recognize that topology and domain theory have been beneficial to each other. As said in [5], domain theory, in a broad sense, is *topology done right*.

Classically, the basic structures in domain theory are directed complete posets (dcpos for short), each of them endowed with a natural topology, called the Scott topology, is a special T_0 space such that every open set is determined by sups of all directed subsets. The category of dcpos with Scott continuous functions can be regarded as a full subcategory of T_0 spaces and is cartesian closed [6, 7, 8, 9]. Note that the category of all T_0 spaces is not cartesian closed. In 2004, Bauer, Birkedal and Scott [10, 11] introduced a notion of an equilogical space, which is a T_0 space with an equivalence relation. They showed that with suitable morphisms the category of equilogical spaces is cartesian closed and can be viewed as admissibility of algebraic lattices. Recently, Battenfeld, Schröder and Simpson [12, 13, 14] introduced an interesting class of topological spaces, named qcb-spaces, which can be identified as a reasonable topological notion of datatype. Most concisely, they are exactly the T_0 topological quotients of countably based spaces, and the category of qcb spaces is countably complete, countably cocomplete, cartesian closed. Even equilogical spaces or qcb-spaces can be used to model several computational phenomena such as higher-order types and computability, the framework of them seems not too satisfactory to replace dcpos and relevant structures in domain theory, because it is difficult to define a natural approximating structure on them. On the other hand, not all computational phenomena can be modeled by dcpos. For example, the unique fully abstract model $\{\mathbb{Q}_\alpha\}$ of hereditarily-sequential finite type functionals for **PCF** is not ω-complete and the topological structure is not the usual Scott topology.

Recently, a special T_0 space, called a directed space is introduced, which is a generality of posets and dcpos endowed with the Scott topologies[15]. The idea is that the priori is not a poset and its existing directed sups, but a T_0 space and its converging directed nets relative to the specialization order. Note that every T_0 space is a poset with respect to its specialization order, and every directed set (relative the specialization order) can regarded as a directed net.

Definition 1.1. [15] Let X be a T_0 space X with its topology $O(X)$.

(1) A subset U of X is called directed-open if for any directed subset $D \subseteq X$ and $x \in X$, D converges to x and $x \in U$ imply $U \cap D \neq \emptyset$.

(2) X is called directed if it is T_0 and all directed-open sets are open.

Let X, Y be two directed spaces. A special product $X \otimes Y$ defined as follows:

- the underlying set of $X \otimes Y$ is $X \times Y$;
- the topology on $X \otimes Y$ is generated by directed limits in the following way: for any $\leq$-directed $D \subseteq X \times Y$ and for $(x, y) \in X \times Y$,

$$D \to (x, y) \text{ in } X \otimes Y \Leftrightarrow \pi_1 D \to x \text{ in } X \text{ and } \pi_2 D \to y \text{ in } Y,$$

  that is, a subset $U \subseteq X \times Y$ is open if and only if $D \cap U \neq \emptyset$ for any $D \to (x, y)$ with $(x, y) \in U$.

Let $O(X \times Y)$ be the product topology of X and Y, let $O(X \otimes Y)$ be the topology defined as above. One always has $O(X \times Y) \subseteq O(X \otimes Y)$, and the specialization order on $X \times Y$ induced by $O(X \times Y)$ is the pointwise order.

A function space $[X \to Y]$ is defined as follows:

- the underlying set of $[X \to Y]$ is Y^X, the set of all continuous functions from X into Y, with the pointwise order;
- the topology on $[X \to Y]$ is generated in the following way: a subset $\mathcal{U} \subseteq Y^X$ is open if for any $\leq$-directed subset $\{f_i : i \in I\} \subseteq Y^X$ and for $f \in U$, $(f_i(x))_{i \in I} \to f(x)$ in Y for any $x \in X$ implies $U \cap \{f_i : i \in I\} \neq \emptyset$.

Set
$\mathbf{TOP_0}$: the category of all T_0 spaces with continuous functions
DTOP: the category of all directed spaces with continuous functions
POSET: the category of all posets endowed with the Scott topologies together with the Scott continuous functions.

DCPO: the category of all dcpos endowed with the Scott topologies together with the Scott continuous functions.

The following result shows that **DTOP** can be viewed as an extended frame of domain theory.

Theorem 1.2. *[16, 17, 18, 19, 20, 21, 22, 15] The category* **DTOP** *has the following properties:*

(1) **DTOP** *is a co-reflective full subcategory of* $\mathbf{Top}_0$.

(2) $\mathbf{DCPO} \subset \mathbf{POSET} \subset \mathbf{DTOP} \subset \mathbf{TOP_0}$.

(3) **DCPO** *is cartesian closed such that*

  - *its product and exponential object are* $X \otimes Y$ *and* $[X \to Y]$ *respectively for all* $X, Y \in$ **DTOP**,

- *the products and exponential objects of* **DCPO** *coincide those in* **DTOP**.

(4) **DTOP** *is closed under Hoare-type', Smyth-type's, Plotkin-type's free algebraic and free topological cone.*

We know that both $\mathbf{TOP_0}$ and **POSET** are not cartesian closed. So here we give the follow conjecture.

Conjecture: **DTOP** is the largest cartesian closed full subcategory of $\mathbf{TOP_0}$ such that containing all c-spaces and all posets with the Scott topologies.

If it is true, it means that **DTOP** is almost the biggest area to extend domain theory in some sense.

References

[1] D. Scott. Outline of a mathematical theory of computation. Oxford University Computing Laboratory, Programming Research Group Oxford, 1970.

[2] D. Scott. Continuous lattices. In *Toposes, algebraic geometry and logic*, pages 97–136. Springer, 1972.

[3] D. Scott. Data types as lattices. *SIAM Journal on Computing*, 5(3):522–587, 1976.

[4] D. S. Scott. Domains for denotational semantics. In *International Colloquium on Automata, Languages, and Programming*, pages 577–610. Springer, 1982.

[5] J. Goubault-Larrecq. *Non-Hausdorff Topology and Domain Theory*. Cambridge University Press, 2013.

[6] S. Abramsky and A. Jung. Domain theory. In *Handbook of Logic in Computer Science*, Vol. III, OUP, 1994, pp. 1–168.

[7] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. Scott. *Continuous Lattices and Domains*. CUP, 2003.

[8] M. Erné. Scott convergence and Scott topology in partially ordered sets II. In B. Banaschewski and R. E. Hoffmann (eds.), *Continuous Lattices*, Lecture Notes in Mathematics, vol. 871, pp. 61–96, Springer, Berlin, 1981.

[9] M. Erné. The ABC of order and topology. In H. Herrlich and H. E. Porst (eds.), *Category Theory at Work*, pp. 57–83, Heldermann, Berlin, 1991.

[10] A. Bauer. A relationship between equilogical spaces and type two effectivity. *Electronic Notes in Theoretical Computer Science*, 45 (2001).

[11] A. Bauer, L. Birkedal, and D. Scott. Equilogical spaces. *Theoretical Computer Science*, 315 (2004), 35–59.

[12] I. Battenfeld. *A Category of Topological Predomains*. Diploma Thesis, TU Darmstadt, 2004.

[13] I. Battenfeld. Comparing free algebras in Topological and Classical Domain Theory. *Theoretical Computer Science*, 411 (2010), 1900–1917.

[14] I. Battenfeld, M. Schröder, and A. Simpson. A Convenient Category of Domains. *Electronic Notes in Theoretical Computer Science*, 172 (2007), 69–99.

[15] Y. Yu and H. Kou. Directed spaces defined through T_0 spaces with specialization order. *Journal of Sichuan University (Natural Science Edition)*, 52 (2015), 217.

[16] Y. Chen. *Related problems in directed spaces and power domains*. PhD Thesis, Sichuan University, 2023.

[17] Y. Chen, H. Kou, Z. Lyv, and X. Xie. A construction of a free dcpo-cone. *Mathematical Structures in Computer Science*, 2024, 1–17.

[18] Y. Chen, H. Kou, and Z. Lyu. Upper power domains of quasi-continuous dcpos. *Theoretical Computer Science*, 1006 (2024), 114663.

[19] Y. Chen, H. Kou, and X. Xie. Upper and Lower power spaces of directed spaces. *Rocky Mountain Journal of Mathematics*, 54 (2024), no. 5, 1299–1314.

[20] Y. Chen, H. Kou, and Z. Lyu. Free algebras over continuous spaces. *Houston Journal of Mathematics*, 50 (2024), no. 4, 751–766.

[21] Y. Chen, H. Kou, and X. Xie. Cartesian closed co-reflective subcategory of topological spaces determined by monotone convergence classes. arXiv:2206.12075.

[22] Y. Chen and H. Kou. S^*-convergence and locally hypercompact spaces. *Journal of Sichuan University (Natural Science Edition)*, 61 (2024), 507.

A few open problems in the theory of continuous valuations

Jean Goubault-Larrecq
Université Paris-Saclay, CNRS, ENS Paris-Saclay,
Laboratoire Méthodes Formelles, 91190 Gif-sur-Yvette, France
jgl@lmf.cnrs.fr

Let $\overline{\mathbb{R}}_+$ be $\mathbb{R}_+ \cup \{\infty\}$, with ∞ largest. A *continuous valuation* on a topological space X is a function ν from the lattice $\mathcal{O}X$ of open subsets of X to $\overline{\mathbb{R}}_+$ that is *strict* ($\nu(\emptyset) = 0$), Scott-continuous (ν is monotonic and for every directed family $(U_i)_{i \in I}$ of open subsets of X, $\nu(\bigcup_{i \in I} U_i) = \sup_{i \in I} \nu(U_i)$) and *modular* (for all $U, V \in \mathcal{O}X$, $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$).

Continuous valuations are close cousins of measures, which were made popular in the semantics of programming languages by Claire Jones [1, 2]. Let $\mathbf{V}X$ denote the set of all continuous valuations on X, with the *stochastic ordering*: $\mu \leq \nu$ if and only if $\mu(U) \leq \nu(U)$ for every $U \in \mathcal{O}X$. With this ordering, $\mathbf{V}X$ is a directed-complete partial order (a *dcpo*), meaning that all directed families have a supremum. Equating every dcpo X with the topological space obtained by equipping it with its Scott topology, this turns $\mathbf{V}$ into an endofunctor on the Cartesian-closed category **Dcpo** of all dcpos and Scott-continuous maps between them, whose action on morphisms is given by $\mathbf{V}f(\nu)(V) \stackrel{\text{def}}{=} \nu(f^{-1}(V))$ for every Scott-continuous map $f \colon X \to Y$ and every $V \in \mathcal{O}Y$. There are analogous endofunctors $\mathbf{V}_1$ of *probability* valuations ($\nu(X) = 1$) and of *subprobability* valuations ($\nu(X) \leq 1$).

It is often desirable to work in full Cartesian-closed subcategories of **Dcpo** with nicer properties, ideally of continuous dcpos. A standard use case is giving semantics of higher-order probabilistic programming languages, where the Cartesian-closed structure serves to interpret the higher-order part and $\mathbf{V}_{\leq 1}$ interprets domains of (sub)probabilistic choice.

Many candidate categories are excluded: $\mathbf{V}$ does not define an endofunctor on the full subcategories of continuous lattices, of bc-domains for example, while the category of continuous dcpos is not Cartesian-closed. The status of two candidates remains open: the category **FS** of *FS-domains* [3], and the category **RB** of *RB-domains*, a.k.a. *retracts of bifinite domains* [4]. **RB** is a full subcategory of **FS**, but it is not known whether **RB** = **FS** or not; this is mentioned in another paper in this volume.

The following questions, already asked by Achim Jung and Regina Tix [5] are therefore natural.

Problem 2.1. Does $\mathbf{V}$ define an endofunctor on **FS**? What about $\mathbf{V}_{\leq 1}$, $\mathbf{V}_1$?

Problem 2.2. Does $\mathbf{V}$ define an endofunctor on **RB**? What about $\mathbf{V}_{\leq 1}$, $\mathbf{V}_1$?

Since RB-domains are bilimits of finite posets (namely, projective limits of diagrams of embedding-projection pairs between finite posets, indexed by a directed poset), and since $\mathbf{V}$ and $\mathbf{V}_{\leq 1}$ preserve bilimits, it would suffice to show that $\mathbf{V}X$, resp. $\mathbf{V}_{\leq 1}X$ are RB-domains for every finite poset X in order to give a positive answer to Problem 2.2. The same remark applies to $\mathbf{V}_1$ provided all RB-domains and all finite posets considered have a least element. A similar observation does not apply to Problem 2.1 unless $\mathbf{RB} = \mathbf{FS}$.

More generally:

Problem 2.3. Is there a full Cartesian-closed category of **Dcpo**, consisting of continuous dcpos, and closed under the action of the $\mathbf{V}$ (resp. $\mathbf{V}_{\leq 1}$, $\mathbf{V}_1$) functor?

The paper [5] is still close to the start of the art today. Among the properties preserved or enforced by the $\mathbf{V}$, $\mathbf{V}_{\leq 1}$, and $\mathbf{V}_1$ endofunctors we find the following.

- For every tree X, $\mathbf{V}X$, $\mathbf{V}_{\leq 1}X$ and $\mathbf{V}_1X$ are bc-domains, and $\mathbf{V}X$ is a continuous lattice; and every continuous lattice is a bc-domain, every bc-domain is an RB-domain, and every RB-domain is an FS-domain. By a *tree*, we mean a set X with an element $*$ called the *root*, and a *predecessor* map $p\colon X \smallsetminus \{*\} \to X$ such that for every $x \in X$, there is a natural number n such that $p^n(x) = *$. X is made a poset by taking $\leq$ to be the smallest ordering that makes $*$ least and $p(x) \leq x$ for every $x \in X \smallsetminus \{*\}$. Special cases of this result are Theorem 13 of [5], which states that $\mathbf{V}_{\leq 1}X$ is an RB-domain for every finite tree X, and Lemma 6.8 of [6], which states that $\mathbf{V}_1X$ is a countably-based bc-domain for every finite tree X. The generalizations we have stated, on general, not finite, trees, are proved by similar arguments. An easy extension is that for every forest X, $\mathbf{V}_{\leq 1}X$ is a bc-domain and $\mathbf{V}X$ is a continuous lattice, where a forest is a disjoint union of trees.

- For every tree X, $\mathbf{V}_{\leq 1}(X^{op})$ is an FS-domain. In other words, applying $\mathbf{V}_{\leq 1}$ to a *reversed* tree produces an FS-domain. This was proved for finite (reversed) trees in [5, Theorem 17], but one can as easily prove it for general (reversed) trees. Jung conjectures that $\mathbf{V}_{\leq 1}Y$ is not an RB-domain already when Y is the three element poset $\{a, b, \bullet\}$ with a and b incomparable and below $\top$ (note that $Y = X^{op}$, where X is a very simple, finite tree.) This would give a negative answer to Problem 2.2, but showing it seems difficult.

- For every finite poset X of height at most 2, $\mathbf{V}_{\leq 1}X$ is an FS-domain. This was stated without proof in the concluding lines of [5], and the proof was made public in [7]. A poset is of *height 2* if it can be written as $\{a_1, \cdots, a_m, b_1, \cdots, b_n\}$ so that the only ordering relations that hold (apart from reflexivity) are of the form $a_i \leq b_j$ for selected pairs of indices i, j.

- For every totally ordered poset X, $\mathbf{V}X$ and $\mathbf{V}_{\leq 1}X$ are continuous lattices, and $\mathbf{V}_1X$, too, if X has a least element, by a result of Mike Mislove [8]. In that case, X is necessarily a continuous poset, with the property that $x < y$ implies $x \ll y$ for all $x, y \in X$, see [9] for this and a simple proof of Mislove's observation.

- For every continuous dcpo X, $\mathbf{V}X$ and $\mathbf{V}_{\leq 1}X$ are continuous dcpos, and so is $\mathbf{V}_1 X$ if X has a least element [2, 10].

- For every Lawson-compact, continuous dcpo X with a least element, $\mathbf{V}_{\leq 1}X$ is also a Lawson-compact, continuous dcpo X with a least element, as shown by Jung and Tix [5, Theorem 19]. This result is somehow subsumed by the results of [11], where $\mathbf{V}_1 X$ and $\mathbf{V}_{\leq 1}X$ are equipped with the weak topology instead of the Scott topology. Those two topologies coincide when X is a continuous dcpo [12, Satz 8.6] or even a quasi-continuous dcpo [13, Corollary 6.2], but we prefer not to talk about the weak topology in this note.

  Every FS-domain is a Lawson-compact, continuous dcpo; in particular, one cannot hope of disproving Problem 2.1 (resp., Problem 2.2) by finding an FS-domain X (resp., an RB-domain or even a finite poset) such that $\mathbf{V}_{\leq 1}X$ would not be Lawson-compact: they simply don't exist.

- For every quasi-continuous dcpo X, $\mathbf{V}X$ and $\mathbf{V}_{\leq 1}X$ are quasi-continuous dcpos, and so is $\mathbf{V}_1 X$ if X has a least element. In short, a quasi-continuous dcpo is a dcpo that is locally finitary compact and sober in its Scott topology. More generally, the same results hold if we only assume that X is a locally finitary compact topological space, with a least element in the case of $\mathbf{V}_1 X$, see [13].

- For every quasi-continuous coherent dcpo X, $\mathbf{V}X$ and $\mathbf{V}_{\leq 1}X$ are quasi-continuous coherent dcpos, and so is $\mathbf{V}_1 X$ if X has a least element. More generally, the same results hold if we only assume that X is a locally finitary compact, coherent topological space, with a least element in the case of $\mathbf{V}_1 X$, see [14]. The quasi-continuous coherent dcpos are the QRB-domains of [6] and also the QFS-domains [15], as shown in [16, 17], and the result of [14] is therefore an extension of [6, Theorem 6.10], which states that for every QRB-domain X with a countable basis (an *ωQRB-domain*), $\mathbf{V}_1 X$ is an ωQRB-domain.

References

[1] Claire Jones and Gordon Plotkin. A probabilistic powerdomain of evaluations. In *Proceedings of the 4th Annual Symposium on Logic in Computer Science*, pages 186–195. IEEE Computer Society Press, 1989.

[2] Claire Jones. *Probabilistic Non-Determinism.* PhD thesis, University of Edinburgh, 1990. Technical Report ECS-LFCS-90-105.

[3] Achim Jung. The classification of continuous domains. In *Proceedings of the Fifth Annual IEEE Symposium on Logic in Computer Science*, pages 35–40. IEEE Computer Society Press, 1990.

[4] Samson Abramsky and Achim Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3, pages 1–168. Oxford University Press, 1994.

[5] Achim Jung and Regina Tix. The troublesome probabilistic powerdomain. In A. Edalat, A. Jung, K. Keimel, and M. Kwiatkowska, editors, *Proc. 3rd Workshop on Computation and Approximation*, volume 13 of *Electronic Lecture Notes in Computer Science*, Amsterdam, The Netherlands, The Netherlands, 1998. Elsevier Science Publishers B. V. 23pp.

[6] Jean Goubault-Larrecq. QRB-domains and the probabilistic powerdomain. *Logical Methods in Computer Science*, 8(1:14), 2012.

[7] Achim Jung. Revisiting the troublesome probabilistic powerdomain. Talk given at Hunan University, March 3rd 2023.

[8] Michael W. Mislove. Domains and stochastic processes. *Theoretical Computer Science*, 807:284–297, 2020.

[9] Jean Goubault-Larrecq. Chains and nested spaces. `https://topology.lmf.cnrs.fr/chains-and-nested-spaces/`, August 2020.

[10] Abbas Edalat. Domain theory and integration. *Theoretical Computer Science*, 151:163–193, 1995.

[11] Achim Jung. Stably compact spaces and the probabilistic powerspace construction. In J. Desharnais and P. Panangaden, editors, *Domain-theoretic Methods in Probabilistic Processes*, volume 87 of *Electronic Lecture Notes in Computer Science*, pages 5–20, Amsterdam, The Netherlands, The Netherlands, 2004. Elsevier Science Publishers B. V. 15pp.

[12] Olaf Kirch. Bereiche und Bewertungen. Diplomarbeit, Technische Hochschule Darmstadt, 1993.

[13] Jean Goubault-Larrecq. Probabilistic powerdomains and quasi-continuous domains. *Topology Proceedings*, 60:1–16, 2022.

[14] Zhenchao Lyu and Hui Kou. The probabilistic powerdomain from a topological viewpoint. *Topology and its Applications*, 237:26–36, 2018.

[15] Gaolin Li and Luoshan Xu. QFS-domains and their Lawson compactness. *Order*, 30:233–248, 2013.

[16] Jean Goubault-Larrecq and Achim Jung. QRB, QFS, and the probabilistic powerdomain. In Bart Jacobs, Bart Silva, and Sam Staton, editors, *Proceedings of the 30th Annual Conference on Mathematical Foundations of Programming Semantics (MFPS XXX)*, pages 170–185, Amsterdam, The Netherlands, The Netherlands, June 2014. Elsevier Science Publishers B. V. Published in Electronic Notes in Theoretical Computer Science.

[17] Jimmie Lawson and Xiaoyong Xi. The equivalence of QRB, QFS, and compactness for quasicontinuous domains. *Order*, 32(2):227–238, July 2015.

A problem connecting infinite-dimensional topology and continuous lattice

Zhong-Qiang Yang
Shantou University, Shantou, Guangdong, China
zqyang@stu.edu.cn

1 A problem connecting infinite-dimensional topology and continuous lattice

Problem 3.1. Let L be a continuous lattice (or a completely distributive lattice (CDL), or a domain,...), ΛL is the topological space L with the Lawson topology. To give (necessary and sufficient; necessary; sufficient) conditions using the order on L such that ΛL is homeomorphic to the Hilbert cube $Q = [0,1]^{\mathbb{N}}$ ($\Lambda L \approx Q$).

We first give some examples as follows:

1. Trivially, Q with the pointwise order is a CDL and $\Lambda Q \approx Q$.

2. Let $\triangle = \{(x_n) \in Q : x_1 \leq x_2 \leq \cdots\}$. As a sublattice of Q, $\triangle$ is also a CDL and $\Lambda\triangle \approx Q$.

3. Let X be a regular T_1 topological space and, $\mathrm{Cld}(X)$ is the family of all non-empty closed sets in X and $\mathrm{Cld}^*(X) = \mathrm{Cld}(X) \cup \{\emptyset\}$ with the inverse inclusion order. Then it is a complete lattice. Moreover, we have the following

(i). $\mathrm{Cld}^*(X)$ is a continuous lattice if and only if X is locally compact;

(ii). The Curtis-Schori-West Hyperspace Theorem: $\Lambda\,\mathrm{Cld}(X) \approx Q$ if and only if X a non-degenerate Peano continuum (connected, locally connected, compact metric space) [1];

(iii). $\Lambda\,\mathrm{Cld}^*(X) \approx Q$ if and only if X is a locally compact, locally connected, separable metrizable space with no compact components [2];

4. Let X be a completely regular T_1 space and let $\mathrm{USC}(X)$ be the set of all upper semicontinuous maps from X to $[0,1]$ with the inversely pointwise order. Then $\mathrm{USC}(X)$ is a complete lattice. Moreover, we have that:

(i). $\mathrm{USC}(X)$ is a continuous lattice if and only if X is locally compact;

(ii). $\Lambda\mathrm{USC}(X) \approx Q$ if and only if X an infinite separable locally compact metrizable space [4];

(iii). $\Lambda F(X) \approx Q$ if X is a non-degenerate convex set in $\mathbb{R}^n$, where $F(X)$ is the set of all fuzzy numbers which supports are conclued in X as a sublattice of $\mathrm{USC}(X)$[6].

Notice that many examples in 3,4 above are not CDLs. For example, $\mathrm{Cld}^*([0,1])$ is a distributive continuous lattice but not a CDL. More examples see [5].

To give an answer for Problem 1.1, we should note the following theorem:

Theorem 3.2 ([3] The Toruńczyk Characterization Theorem for the Hilbert Cube)**.** *A metrizable space X is homeomorphic to the Hilbert cube Q if and only if it is a compact absolute retract with the disjoint-cells property.*

Here, a metrizable space X is called an *absolute retract* if for each metrizable space Y including X as a closed subspace, there exists a continuous map $r : Y \to X$ such that $r|_X = \mathrm{id}_X$. A compact metric space (X, d) is said to have *disjoint cells property* if for each n and for each $f : [0,1]^n \times \{1,2\} \to X$, and for each $\varepsilon > 0$, there exists a continuous map $g : [0,1]^n \times \{1,2\} \to X$ such that $d(f,g) < \varepsilon$ and $g([0,1]^n \times \{1\}) \cap g([0,1]^n \times \{2\}) = \emptyset$.

We hope to give an answer for Problem 1.1 such that the examples in 3,4 are its corollaries.

References

[1] D.W. Curtis and R.M. Schori, *Hyperspaces of Peano continua are Hilbert cube*, Fund. Math., **101**(1978), 19–38.

[2] K. Sakai and Z. Yang, *Hyperspaces of non-compact metrizable spaces which are homeomorphic to the Hilbert cube*, Topology Appl., **127**(2003), 331–342.

[3] H. Toruńczyk, *On CE-images of the Hilbert cube and characterizations of Q-manifolds*, Fund. Math., **106**(1980), 31–40.

[4] Z. Yang, *The hyperspace of the regiond of all lattice continuous and its Hilbert cube compactification*, Sci. China Ser. A, **48**:4(2005), 469–484.

[5] Z. Yang, H. Yang and E. Wu, Topological Structures of Function Spaces, Beijing World Publishing Corporation, Ltd., 2025.

[6] Z. Yang and L. Zhang, *The topological structure of fuzzy sets with endograph metric*, Fuzzy Sets Syst., **160**(2009), 2937–2946.

Amadio–Curien Problem

Xiaoyong Xi
School of Mathematics and Statistics, Jiangsu Normal University,
Xuzhou, Jiangsu, China
xixiaoyong@jsnu.edu.cn

The study of Cartesian closed categories received considerable attention in the study of domains. Let **CONT**, **ALG** be the categories of continuous domains, algebraic domains with Scott-continuous mappings respectively. The Cartesian closedness of full subcategories of **CONT**, **ALG** is investigated by Smyth [1], Jung [2, 3] and others. Jung [2, 3] systematically studied the Cartesian closed maximal full subcategories of **CONT**, **ALG** and proved that there exist exactly four Cartesian closed maximal full subcategories of **CONT**, **ALG**, respectively. On the other hand, motivated from the study of the sequential computation, Berry [4] introduced the stable mappings and the category of dI-domains. The stable domain theory played significant roles in linear logic, concurrency, polymorphism in PCF, and object-oriented programming and was investigated by many authors. Zhang [5] started the investigation of the question of "largest Cartesian closed categories of stable domains" and showed that Berry's category of dI-domains is the largest Cartesian closed subcategory inside the category of Scott domains (which are bounded complete) with stable functions. Amadio [6] and Droste [7] generalised the stable domain theory to the setting of algebraic L-domains and introduced an interesting new category called stable bifinite domains, which is similar to the Plotkin's SFP domains. Amadio [6] and Droste [7] showed that the category of stable bifinite domains (**SB** for short) is Cartesian closed. Amadio [6] raised the question of whether the category **SB** of Amadio–Droste [6, 7] is the largest Cartesian closed full sub-category of the category of ω-algebraic meet-cpos with stable functions (**SAM** for short). Zhang and Jiang [8] considered the stable function space $[D \to_s D]$ and showed that, for any ω-algebraic meet-cpo D, if $[D \to_s D]$ (with stable order) satisfies property M, then D is finitary (i.e., each compact element dominates only finitely many elements). In the sense of stable order, it seems difficulty to characterize the (mub,down)-closure generated by finitely many stable mappings in $[D \to_s D]$. Joint with Kou and Yang [9] , we consider the (higher-order) function spaces built from D and the diamond lattice M_3 (the classical nondistributive lattice) and obtained the following result.

Theorem 4.1. *For any ω-algebraic meet-cpo domain D, if $[D \to_s M_3]$ and $[[D \to_s M_3] \to_s [D \to_s M_3]]$ are ω-algebraic, then*
(1) D is finitary;
(2) if $[[D \to_s M_3] \to_s [D \to_s M_3]]$ is finitary, then D is stable bifinite.

So the category **SB** is a maximal Cartesian closed full subcategory of **SAM**. This gives a partial solution to the problem. Smyth[1] proved that the category of SFP domains, which is also called bifinite domain introduced by Plotkin and systematically studied by Ahim Jung, is the largest Cartesian closed subcategory of the category of ω-algebraic domains. We give a similar parallel result for the stable case and then give the (almost complete) classification of largest Cartesian closed subcategory (**CCC** for short) of stable algebraic domains (see Fig.1). Note that in Fig.1, the hollow dots indicate that the category is not Cartesian closed, while the solid black dots indicate that it is a Cartesian closed subcategory.

The diamond lattice M_3 plays an important role in the proof. Suppose that **C** is a Cartesian closed full subcategory of **SAM**. If **C** contains an object which has a copy of M_3, then **C** will be a subcategory of **SB**. But an ω-algebraic meet-cpo may not contain such a copy (see Fig. 2). So our method can not solve the problem completely. But for the cpo D denoted by Fig. 2, $[D \to_s D]$ fails property M and therefore $[[D \to_s D] \to_s [D \to_w D]]$ is not ω-algebraic. In view of this example and many other examples, we believe the problem is true. But the proof to solve it completely may be complicated. The following problem posed by Amadio, Curien and Droste is still open.

Problem: Is the category of countably based stable bifinite domains the largest cartesian closed full subcategory within stable ω-algebraic L-domains?

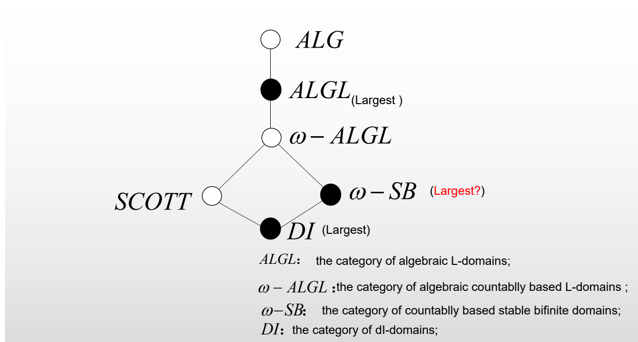

Figure 1: The hierarchy of CCC of stable algebraic domains

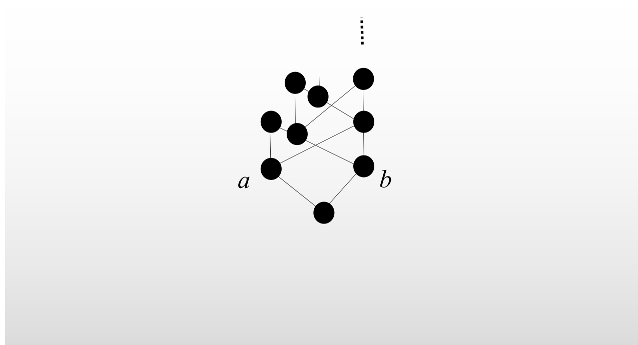

Figure 2: The function space does not satisfy Property **M**

References

[1] M. B. Smyth. The largest Cartesian closed category of domains. *Theoretical Computer Science*, 127 (1983), 109–119.

[2] S. Abramsky and A. Jung. Domain Theory. In S. Abramsky et al. (eds.), *Handbook of Logic in Computer Science*, vol. 3, Clarendon Press, 1995.

[3] A. Jung. *Cartesian Closed Categories of Domains*. CWI Tracts, vol. 66, 1989.

[4] G. Berry. Stable models of typed λ-calculi. In *Lecture Notes in Computer Science*, vol. 62, 1978, pp. 72–88.

[5] G.-Q. Zhang. The largest Cartesian closed category of stable domains. *Theoretical Computer Science*, 166 (1996), 203–219.

[6] R. M. Amadio. Bifinite domains: stable case. In *Lecture Notes in Computer Science*, vol. 530, 1991, pp. 16–33.

[7] M. Droste. On stable domains. *Theoretical Computer Science*, 111 (1993), 89–101.

[8] G.-Q. Zhang and Y. Jiang. On a problem of Amadio and Curien: the finite antichain condition. *Information and Computation*, 202 (2005), 87–103.

[9] Xiaoyong Xi, J. Yang, and H. Kou. A partial solution to an open problem of Amadio and Curien. *Information and Computation*, 260 (2018), 126–134.

C_σ-determined dcpos

Dongsheng Zhao
Mathematics and Mathematical Education, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore
dongsheng.zhao@nie.edu.sg

For any topological space X, let $\Gamma(X)$ be the complete lattice of all closed subsets of X. If X and Y are sober spaces, then X is homeomorphic to Y if and only if the lattices $\Gamma(X)$ and $\Gamma(Y)$ are isomorphic.

For any dcpo P, let $C_\sigma(P)$ be the complete lattice of all Scott closed subsets of P.

Let L be a domain (continuous dcpo). If P is a dcpo such that the complete lattices $C_\sigma(L)$ and $C_\sigma(P)$, then P is also a domain. Now the Scott spaces ΣL and ΣP are sober spaces and their closed set lattices are isomorphic, hence $\Sigma\, L$ is homeomorphic to $\Sigma\, P$, which further implies that L is homeomorphic to P.

Definition 5.1. A dcpo L is called C_σ-determined, if for any dcpo P, L is isomorphic to P whenever $C_\sigma(L)$ and $C_\sigma(P)$ are isomorphic.

An example was constructed in [1] to show that not every dcpo is C_σ-determined. The other genera C_σ-determined dcpos are considered in [2, 3, 4].

The following are some open problems on C_σ-determined dcpos.

Problem 1. Can we establish a necessary condition for C_σ-determined dcpos?

Problem 2. Is every complete lattice C_σ-determined?

Before answering the problem 2, one may try the following two more specific problems.

Problem 3. Is every countable complete lattice C_σ-determined?

The most challenging problem might be the following one.

Problem 4. Can one establish a complete characterisation for C_σ-dcpos?

References

[1] W. Ho, J. Goubault-Larrecq, A. Jung, and X. Xi. The Ho-Zhao problem. *Logical Methods in Computer Science*, 14 (2018), 1:1–19.

[2] L. Xu and D. Zhao. C_σ-unique dcpos and non-maximality of the class of dominated dcpos regarding Γ-faithfulness. In *Proceedings of ISDT 2019*, Electronic Notes in Theoretical Computer Science, 345, pp. 249–260, Elsevier B. V., Amsterdam, 2019.

[3] D. Zhao and L. Xu. Uniqueness of directed complete poset based on Scott closed set lattices. *Logical Methods in Computer Science*, 14(2:10) (2018), 1–12.

[4] D. Zhao. Directed complete posets determined by Scott closed set lattices and related work. *Questions and Answers in General Topology*, 40 (2022), 91–106.

Ten open problems about semitopological cones and semitopological barycentric algebras

Jean Goubault-Larrecq
Université Paris-Saclay, CNRS, ENS Paris-Saclay,
Laboratoire Méthodes Formelles, 91190 Gif-sur-Yvette, France
jgl@lmf.cnrs.fr

A *cone* $\mathfrak{C}$ is like a real vector space, except that only scalar multiplication by *non-negative* real numbers is allowed, and the axioms that require subtraction or multiplication by negative numbers are removed. A cone is *semitopological* if and only if addition $+ \colon \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$ and scalar multiplication $\cdot \colon \mathbb{R}_+ \times \mathfrak{C} \to \mathfrak{C}$ are separately continuous, where $\mathbb{R}_+$ is given the Scott topology of its usual ordering. Then scalar multiplication is jointly continuous. $\mathfrak{C}$ is *topological* if and only if $+$ is jointly continuous as well. These were studied by Klaus Keimel [1].

A *barycentric algebra* is an abstraction of the notion of convex set. This is a set $\mathfrak{B}$ with a ternary operation $(x, a, y) \in \mathfrak{B} \times [0, 1] \times \mathfrak{B} \mapsto x +_a y \in \mathfrak{B}$ satisfying the laws one would expect from the map $(x, a, y) \mapsto a \cdot x + (1 - a) \cdot y$ on a cone $\mathfrak{C}$ or a real vector space. $\mathfrak{B}$ is *semitopological* if the map $(x, a, y) \mapsto x +_a y$ is separately continuous, where $[0, 1]$ is given its usual metric topology; it is *quasitopological* if and only the map $(x, a) \mapsto x +_a y$ is jointly continuous for every fixed $y \in \mathfrak{B}$; it is *topological* if and only if $(x, a, y) \mapsto x +_a y$ is jointly continuous. See [2] for their theory. The same paper lists a few open problems, which we will summarize, and expand upon slightly, here.

One example of a topological cone is the space $\mathfrak{M}X$ of all Borel measures on a topological space X, or the space $\mathbf{V}X$ of continuous valuations, with the weak topology, and the obvious algebraic operations. This does not embed into any real vector space, because any cone that embeds in a vector space is cancellative ($x + z = y + z$ implies $x = y$), and the fact that measures and valuations can assume the value ∞ contradicts cancellativity. The subspaces $\mathfrak{M}_1 X$ of probability measures on X, or $\mathbf{V}_1 X$ of probability valuations on X, are topological barycentric algebras. The spaces $\mathfrak{M}_{\leq 1} X$ of subprobability measures and $\mathbf{V}_{\leq 1} X$ of subprobability valuations are *pointed* topological algebras. A pointed (semi-, quasi-)topological barycentric algebra is one with a distinguished element $\perp$ (the *point*) that is least in the specialization preordering: in the cases of $\mathfrak{M}_{\leq 1} X$ and $\mathbf{V}_{\leq 1} X$, this is the zero measure and the zero valuation.

Every topological cone is semitopological. For a barycentric algebra, we have the implications topological $\Rightarrow$ quasitopological $\Rightarrow$ semitopological. Every semitopological cone, seen as a semitopological barycentric algebra, is quasitopological, but we may wonder whether this generalizes to barycentric algebras.

Problem 6.1. Is every semitopological barycentric algebra actually quasitopological? Is every pointed semitopological barycentric algebra quasitopological?

For every topological space X, let $\mathcal{L}X$ denote the semitopological cone of lower semicontinuous maps from X to $\overline{\mathbb{R}}_+$, with the Scott topology of the pointwise ordering. It is known that $\mathcal{L}X$ is topological if X is core-compact [3, Example 3.5].

Problem 6.2. Is there a space X for which $\mathcal{L}X$ is not a topological cone? Can we characterize those spaces X such that $\mathcal{L}X$ is topological?

There is a free semitopological cone $\mathrm{cone}(\mathfrak{B})$ over any semitopological cone $\mathfrak{B}$, a free pointed semitopological barycentric algebra $\mathrm{cone}_{\leq 1}(\mathfrak{B})$ over any semitopological cone $\mathfrak{B}$, and a free semitopological cone $\mathrm{tscope}_\alpha(\mathfrak{B})$ over any pointed semitopological barycentric algebra $\mathfrak{B}$ (where $\alpha \in]0,1[$ is arbitrary: we obtain isomorphic semitopological cones $\mathrm{tscope}_\alpha(\mathfrak{B})$ whatever α is). If $\mathfrak{B}$ is topological, then $\mathrm{cone}(\mathfrak{B})$ is topological, too.

Problem 6.3. Given a pointed topological barycentric algebra $\mathfrak{B}$, is $\mathrm{tscope}_\alpha(\mathfrak{B})$ a topological cone?

We say that a semitopological barycentric algebra $\mathfrak{B}$ is *embeddable* if and only if there is an affine topological embedding of $\mathfrak{B}$ into some semitopological cone. (A map f is *affine* if and only if it satisfies $f(x +_a y) = f(x) +_a f(y)$.) It is equivalent to require that the canonical map $\eta^{\mathrm{cone}}_{\mathfrak{B}} \colon \mathfrak{B} \to \mathrm{cone}(\mathfrak{B})$ be a topological embedding (more explicitly, $\eta^{\mathrm{cone}}_{\mathfrak{B}}$ is the unit of the adjunction underlying the free construction $\mathrm{cone}(\mathfrak{B})$). It is known that not all semitopological barycentric algebras are embeddable. The simplest counterexample is $]-\infty, 0]$ with the Scott topology and the obvious operation $x +_a y$ defined as $a \cdot x + (1-a) \cdot y$; it is even a topological barycentric algebra [2, Example 3.24].

Perhaps surprisingly, all *pointed* semitopological barycentric algebras are embeddable, but the embedding may fail to preserve the point. A *strict* map is one that preserves the point, and a *linear* map is a strict, affine map; this coincides with the usual notion of linearity on cones. A pointed semitopological barycentric algebra $\mathfrak{B}$ is *strictly embeddable* if and only if there is a linear topological embedding of $\mathfrak{B}$ into some semitopological cone; equivalently, if the canonical map $\eta^{\mathrm{tscope},\alpha}_{\mathfrak{B}} \colon \mathfrak{B} \to \mathrm{tscope}_\alpha(\mathfrak{B})$ is a topological embedding, if and only if the map $x \mapsto \alpha \cdot x$ is a topological embedding of $\mathfrak{B}$ into itself for some (equivalently, for every) $\alpha \in]0,1[$. There is a pretty complicated example of a non-strictly embeddable pointed semitopological barycentric algebra [2, Example 4.28]; we suspect that it is not topological.

Problem 6.4. Is every pointed topological barycentric algebra strictly embeddable? Same question with "quasitopological" instead of "topological".

A semitopological barycentric algebra (or cone) $\mathfrak{B}$ is *locally affine* (resp. *locally linear*, when $\mathfrak{B}$ is pointed or a cone) if and only if its topology has a subbase of open sets of the form $\Lambda^{-1}(]1,\infty])$, where Λ ranges over the affine (resp. linear) lower semicontinuous maps from $\mathfrak{B}$ to $\overline{\mathbb{R}}_+$; $\overline{\mathbb{R}}_+$ is $\mathbb{R}_+$ plus a point ∞ larger than all others. "Locally affine" and "locally linear" really mean the same thing on semitopological pointed barycentric algebras and cones, but "locally linear" is the standard notion on cones, and would make no sense on non-pointed semitopological barycentric algebras. Every locally affine semitopological barycentric algebra is topological.

A semitopological barycentric algebra (or cone) $\mathfrak{B}$ is *locally convex* if and only if every point has a base of convex open neighborhoods. It is *weakly locally convex* if and only if every point has a base of convex neighborhoods. We have the implications locally affine $\Rightarrow$ locally convex $\Rightarrow$ weakly locally convex. $]-\infty, 0]$ is locally convex but not locally affine, because its only affine lower semicontinuous maps to $\overline{\mathbb{R}}_+$ are constant.

Problem 6.5. Is weak local convexity the same notion as local convexity on semitopological cones? on semitopological pointed barycentric algebras? on semitopological barycentric algebras?

Not every semitopological, or even topological, barycentric algebra is weakly locally convex: there is a counterexample, inspired by one due to Tychonoff in the realm of locally convex real vector spaces [2, Example 6.44]. A more complex counterexample shows that not every semitopological cone is weakly locally convex [2, Example 6.45].

Problem 6.6. Is there any topological (not just semitopological) cone that is not weakly locally convex? or at least not locally convex?

Problem 6.7. For a semitopological barycentric algebra $\mathfrak{B}$, is it true that if $\mathfrak{B}$ is locally convex (resp. weakly locally convex), then $\mathrm{cone}(\mathfrak{B})$ is, or $\mathrm{cone}_{\leq 1}(\mathfrak{B})$ is, or $\mathrm{tscope}_\alpha(\mathfrak{B})$ is, assuming $\mathfrak{B}$ pointed in the last case? What about the converse statements?

Let us say that a semitopological barycentric algebra is *lc-embeddable* if and only if it embeds through an affine topological embedding into a locally convex semitopological cone. Every locally affine T_0 semitopological barycentric algebra is lc-embeddable [2, Remark 10.16].

Problem 6.8. Can we characterize the lc-embeddable semitopological barycentric algebras? Are those the semitopological barycentric algebras $\mathfrak{B}$ such that $\mathrm{cone}(\mathfrak{B})$ is locally convex? (I doubt it). Same questions for pointed barycentric algebras, with tscope_α instead of cone.

We do not know of any space X for which $\mathcal{L}X$ would not be locally convex. One can show that $\mathcal{L}\mathbb{Q}$ is not locally linear [2, Example 6.40], and that $\mathcal{L}X$ is locally linear, hence locally convex, for every space X whose sobrification is $\odot$-consonant, see [3, Example 3.14].

Problem 6.9. Is $\mathcal{L}X$ locally convex for every space X? or at least weakly locally convex?

A semitopological barycentric algebra $\mathfrak{B}$ is *consistent* if and only if for all open subsets U and V of $\mathfrak{B}$, for every open subset I of $[0,1]$, $\Uparrow(U +_I V)$ is open in $\mathfrak{B}$, where $U +_I V \stackrel{\mathrm{def}}{=} \{x +_a y \mid x \in U, y \in V, a \in I\}$. It is *strongly consistent* if and only if for all open subsets U and V of $\mathfrak{B}$, for every $a \in [0,1]$, $\Uparrow(U +_a V)$ is open in $\mathfrak{B}$, where $U +_a V \stackrel{\mathrm{def}}{=} \{x +_a y \mid x \in U, y \in V\}$. Strong consistency implies consistency, and the two notions are equivalent on topological barycentric algebras, and also on semitopological cones, where they coincide with Keimel's requirement that addition be almost open [1]. For every space X, $\mathbf{V}X$ and $\mathbf{V}_{\leq 1}X$ are strongly consistent, while $\mathbf{V}_1X$ is if X has a least element in its specialization preordering. It is noteworthy that for a consistent semitopological barycentric algebra, weak local convexity and local convexity coincide [2, Corollary 7.10].

Problem 6.10. Is every consistent semitopological barycentric algebra strongly consistent?

References

[1] Klaus Keimel. Topological cones: Functional analysis in a T_0-setting. *Semigroup Forum*, 77(1):109–142, 2008.

[2] Jean Goubault-Larrecq. Semitopological barycentric algebras. arXiv report 2512.12865, version 5 (first version: December 2025), June 2026. Submitted to a journal.

[3] Jean Goubault-Larrecq and Xiaodong Jia. Algebras of the extended probabilistic powerdomain monad. *Electronic Notes in Theoretical Computer Science*, 345:37–61, 2019. Proc. 8th International Symposium on Domain Theory (ISDT'19).

The “Collatz Problem” of domain theory: is FS = RB?

Weng Kin Ho
National Institute of Education,
Nanyang Technological University, Singapore
wengkin.ho@nie.edu.sg

Abstract

We survey the open problem of whether the class of finitely separated domains coincides with the class of retracts of bifinite domains. The problem is easy to state but remains unresolved, and is widely regarded as a “Collatz-type” question in domain theory. We summarize the definitions, known inclusions, key examples, and current limitations.

1 Introduction

Domain theory provides mathematical models of computation via directed complete partial orders (dcpos) and Scott-continuous maps. Among continuous domains, several finitary subclasses have been proposed to capture computationally meaningful approximations.

Two such classes are:

- **RB**: retracts of bifinite (i.e. algebraic and coherent) domains;
- **FS**: finitely separated domains.

The central open problem is:

Problem (FS = RB?). Does **FS** = **RB**?

This question has been informally compared to the Collatz conjecture: simple to formulate, yet resistant to existing techniques.

2 Preliminaries

Definition 7.1. A *dcpo* is a partially ordered set $(P, \leq)$ in which every directed subset has a supremum. A dcpo is *pointed* if it has a least element $\perp$.

Definition 7.2. For a dcpo P, the *way-below relation* $\ll$ is defined by

$$y \ll x \iff \forall D \subseteq P \text{ directed}, \bigvee D \geq x \Rightarrow \exists d \in D,\ d \geq y.$$

Definition 7.3. A dcpo P is a *continuous domain* if for every $x \in P$, the set $\{y \mid y \ll x\}$ is directed with supremum x.

Definition 7.4. An element x is *compact* if $x \ll x$. A domain is *algebraic* if every element is the supremum of compact elements below it.

A fundamental structural result is:

Theorem 7.5. *Every continuous domain is a retract of an algebraic domain.*

3 The Classes RB and FS

Definition 7.6. A domain P is in **RB** if it is a Scott-continuous retract of a bifinite domain.

Here bifinite domains are algebraic domains satisfying an additional finiteness condition (coherence plus finitary basis), ensuring good computational behavior.

Definition 7.7. A domain P is *finitely separated* (FS) if for any finite set $F \subseteq P$ and any $x \not\leq \bigvee F$, there exists a Scott-continuous map to a finite poset separating x from F.

Intuitively, FS domains admit finite observations that distinguish elements.

4 Known Results

The relationship between these classes is only partially understood.

Proposition 7.8. $\mathbf{RB} \subseteq \mathbf{FS} \subseteq \mathbf{Dom}$.

The inclusion $\mathbf{RB} \subseteq \mathbf{FS}$ follows from the finitary structure inherited from bifinite domains. The inclusion $\mathbf{FS} \subseteq \mathbf{Dom}$ is immediate.

Despite substantial effort, neither of the following has been established:

- $\mathbf{FS} \subseteq \mathbf{RB}$;
- existence of a domain in $\mathbf{FS} \setminus \mathbf{RB}$.

Non-algebraic FS domains

There exist FS domains that are not algebraic. A key example is the Lawson disc domain.

Definition 7.9. Let L be the set of closed discs in $\mathbb{R}^2$, together with $\mathbb{R}^2$, ordered by reverse inclusion:

$$D \leq E \iff D \supseteq E.$$

Proposition 7.10. *The dcpo $(L, \leq)$ is a continuous domain but not algebraic.*

The way-below relation admits a geometric characterization via interior containment. This domain has very few compact elements, yet still admits a basis via approximation.

Recent observations show:

Proposition 7.11. *The Lawson disc domain is finitely separated.*

This provides a nontrivial FS example beyond the algebraic setting, sharpening the FS–RB comparison.

5 Discussion

The problem $\mathbf{FS} = \mathbf{RB}$ remains open due to several obstacles:

- **Intrinsic vs. extrinsic definitions:** FS is defined via separation properties, while RB is defined via constructions (retracts).
- **Lack of representation theorems:** No general method converts FS domains into retracts of bifinite domains.
- **Absence of counterexamples:** No domain is known to lie in $\mathbf{FS} \setminus \mathbf{RB}$.

It is conceivable that new structural or categorical tools are required to resolve the problem.

6 Conclusion

The question $\mathbf{FS} = \mathbf{RB}$ compares two fundamental finitary notions in domain theory. A positive answer would unify intrinsic and representation-based approaches to finiteness; a negative answer would reveal a strictly larger class of finitely well-behaved domains.

At present, the problem remains a central open question in the field.

References

[1] S. Abramsky and A. Jung, *Domain Theory*, Handbook of Logic in Computer Science, 1994.

[2] G. Gierz, K. Hofmann, K. Keimel, J. Lawson, M. Mislove, and D. Scott, *Continuous Lattices and Domains*, Cambridge University Press, 2003.

[3] J. D. Lawson, The structure of continuous lattices, Various papers.

[4] W. K. Ho, *The "Collatz Problem" of Domain Theory: Is FS = RB?*, Seminar talk, Tianyuan Mathematics Research Centre, 2025.

The Fubini–Tonelli equations for continuous valuations on DCPO

Xiaodong Jia
School of Mathematics, Hunan University,
Changsha, Hunan 410082, China
jiaxiaodong@hnu.edu.cn

Let $\overline{\mathbb{R}}_+ = [0, \infty]$ be the non-negative extended reals, with the arithmetic extended by $r \cdot \infty = 0$ if $r = 0$, and $r \cdot \infty = \infty$ otherwise; for addition, $r + \infty = \infty + r = \infty$ always holds. Let X be a topological space and $\mathcal{O}X$ denote the set of all open sets of X. A (extended) continuous valuation μ on X is a function $\mu \colon \mathcal{O}X \to \overline{\mathbb{R}}_+$ that is

1. strict – $\mu(\emptyset) = 0$,
2. monotone – $\mu(U) \leq \mu(V)$ if $U \subset V$,
3. modular – $\mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V)$,
4. Scott-continuous – $\mu(\bigcup_{i \in I} U_i) = \sup_{i \in I} \mu(U_i)$ for any directed family of open sets $U_i, i \in I$.

A continuous valuation μ on X is called bounded if $\mu(X) < \infty$. The set of all continuous valuations defined on X is denoted by $\mathcal{V}X$. Let $f \colon X \to \overline{\mathbb{R}}_+$ be a lower semi-continuous map, that is, $f^{-1}(t, \infty]$ is open for each $t \in \mathbb{R}$. Such an f is just a continuous map when $\overline{\mathbb{R}}_+$ is equipped with the Scott topology, the topology consisting of all upper sets that are also inaccessible by suprema of directed subsets. For a given continuous valuation μ on X, one could define an integral of f against μ à la Choquet integral:

$$\int_X f(x)\, d\mu = \int_0^\infty \mu(f^{-1}(t, \infty])\, dt.$$

As the function $t \mapsto \mu(f^{-1}(t, \infty])$ is decreasing, the Riemann integral on the right side of the equation makes sense. Now suppose we have two topological spaces X and Y, and μ, ν be two continuous valuations defined on them, respectively. For a lower semi-continuous map $f \colon X \times Y \to \overline{\mathbb{R}}_+$, it was proved in [1, Theorem 3.17] (the bounded case) and in [2, Proposition 4.1] that the following nested integrals agree.

Theorem 8.1 (Fubini-Tonelli for continuous valuations on **Top**). *Let X and Y be two topological spaces, and $\mu \in \mathcal{V}X$, $\nu \in \mathcal{V}Y$. There is a unique product valuation $\mu \times \nu$ on $X \times Y$ such that for every $U \in \mathcal{O}X$ and for every $V \in \mathcal{O}Y$, $(\mu \times \nu)(U \times V) = \mu(U) \cdot \nu(V)$. For every lower semi-continuous function $f \colon X \times Y \to \overline{\mathbb{R}}_+$,*

$$\int_{(x,y) \in X \times Y} f(x, y)\, d(\mu \times \nu) = \int_{x \in X} \left(\int_{y \in Y} f(x, y)\, d\nu \right) d\mu$$

$$= \int_{y\in Y} \left(\int_{x\in X} f(x,y)\, d\,\mu \right) d\,\nu.$$

Let P be a directed complete poset (dcpo), that is, every directed subset of P has a supremum. Every dcpo P can be seen as a topological space P_σ with its Scott topology σP. Let us denote by $\mathcal{V}P$ the set of all continuous valuations defined on P_σ. As we have the Fubini-Tonelli equations for continuous valuations on topological spaces, one would expect that we would obtain a Fubini-Tonelli theorem for Scott-continuous maps and continuous valuations on dcpos, as an special case of Theorem 8.1. But this is not the case.

Question 8.2 (The Fubini-Tonelli problems on dcpos)**.** *Let P and Q be two dcpos, and $\mu \in \mathcal{V}P$, $\nu \in \mathcal{V}Q$. For a Scott-continuous function $f\colon P\times Q \to \overline{\mathbb{R}}_+$, do we have that*

$$\int_{x\in P} \left(\int_{y\in Q} f(x,y)\, d\,\nu \right) d\,\mu = \int_{y\in Q} \left(\int_{x\in P} f(x,y)\, d\,\mu \right) d\,\nu \text{ ?} \tag{1}$$

The reason that we do not know an answer to Question 8.2 is subtle: products of dcpos do not usually coincide with products of their topological counterparts. Explicitly, the topology on $(P\times Q)_\sigma$ (where $\times$ is dcpo product) is finer, and in general strictly finer, than the product topology on $P_\sigma \times Q_\sigma$. (See [3, Exercise II-4.26] for an example where it is strictly finer.) As a consequence, there are more, and generally, strictly more Scott-continuous maps from $P\times Q$ to $\overline{\mathbb{R}}_+$ than continuous maps from the topological product $P_\sigma\times Q_\sigma$ to $\overline{\mathbb{R}}_+$. The Fubini-Tonelli formula holds for functions of the second, smaller class, but it is unknown whether it holds for the first kind of functions.

This observation indicates that we *do* have that Equation 1 holds if the Scott and product topologies coincide on $P\times Q$, as Theorem 8.1 applies. Below we list several sufficient conditions ensuring coincidence.

1. If either P or Q is *core-compact*, i.e., either σP or σQ is a continuous lattice in the inclusion order, then $(P\times Q)_\sigma = P_\sigma \times Q_\sigma$, see [3, Theorem II-4.13], and Equation 1 holds.

2. If both P_σ and Q_σ are first countable, then $(P\times Q)_\sigma = P_\sigma\times Q_\sigma$. This is a result discovered by Matthew de Brecht, and for an argument see Jean Goubault-Larrecq's blog post [4]. Hence Equation 1 holds.

3. If both P and Q only have countably many non-trivial ideals, then $(P\times Q)_\sigma = P_\sigma \times Q_\sigma$ [5, Lemma 4.1], and Equation 1 holds.

4. If both P and Q are lc_ω dcpos, then $(P\times Q)_\sigma = P_\sigma\times Q_\sigma$ [6, Theorem 4.10], and Equation 1 holds.

The aforementioned are situations where we put restrictions on P and Q, hence on Scott-continuous maps from $P\times Q$ to $\overline{\mathbb{R}}_+$. The Equation 1 also holds when we put certain restrictions on continuous valuations.

- A continuous valuation μ on a topological space X is called *point-continuous* if for every open set U with $\mu(U) > r$, there exist a finite subset F of U such that for any open subset V containing F, $\mu(V) > r$.
- Let X be a space and $\mu \in \mathcal{V}X$. For a continuous map $f\colon X \to P_\sigma$, the push-forward valuation $f[\mu]$ of μ along f, defined on P, is such that $f[\mu](U) = \mu(f^{-1}(U))$.

We have the following positive results.

5. If either μ or ν in Question 8.2 is point-continuous, Equation 1 holds. See [7, Theorem 21].
6. If μ (or ν) in Question 8.2 is of the form $f[\mu']$, for some μ', defined on a *core-compact* space X, and some continuous $f\colon X \to P$ (or to Q), then Equation 1 holds. See [8, Theorem].

A related question is whether Item 5 can be subsumed by Item 6.

Question 8.3. *Let μ be a point-continuous valuation on a dcpo P. Is μ of the form $f[\mu']$, for some μ' defined on a core-compact space X and some continuous $f\colon X \to P$?*

References

[1] Claire Jones. *Probabilistic Non-Determinism*. PhD thesis, University of Edinburgh, Edinburgh, 1990. Also published as Technical Report No. CST-63-90.

[2] Jean Goubault-Larrecq, Xiaodong Jia, and Clément Théron. A domain-theoretic approach to statistical programming languages. *J. ACM*, 70(5), October 2023.

[3] Gerhard Gierz, Karl H. Hofmann, Klaus Keimel, Jimmie D. Lawson, Michael Mislove, and Dana S. Scott. *Continuous Lattices and Domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2003.

[4] Jean Goubault-Larrecq. On countability: the compact completed sequence, a note by Matthew de Brecht. Available at `https://topology.lmf.cnrs.fr/on-countability-the-compact-completed-sequence/`.

[5] Hualin Miao, Xiaoyong Xi, Qingguo Li, and Dongsheng Zhao. Not every countable complete distributive lattice is sober. *Mathematical Structures in Computer Science*, 33(9):809–831, 2023.

[6] Jimmie Lawson and Xiaoquan Xu. Spaces determined by countably many locally compact subspaces. *Topology and its Applications*, 354:108973, 2024.

[7] Xiaodong Jia, Bert Lindenhovius, Michael Mislove, and Vladimir Zamdzhiev. Commutative monads for probabilistic programming languages. In *Proc. 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS'21)*, pages 1–14, Rome, Italy, 2021. IEEE Press. Article 19.

[8] Xiaodong Jia, Michael Mislove, and Vladimir Zamdzhiev. The central valuations monad. In *9th Conference on Algebra and Coalgebra in Computer Science (CALCO 2021)*, volume 211 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 18:1–18:5. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021.

Two problems related to quasicontinuous domains

Xiaoquan Xu
School of Mathematics and Statistics, Minnan Normal University,
Zhangzhou, Fujian, China
xiqxu2002@163.com

1 Two problems related to quasicontinuous domains

For a dcpo P and $A, B \subseteq P$, we say A is *way below* B, written $A \ll B$, if for each $D \in \mathcal{D}(P)$, $\vee D \in \uparrow B$ implies $D \cap \uparrow A \neq \emptyset$. For $B = \{x\}$, a singleton, $A \ll B$ is written $A \ll x$ for short. For $x \in P$, let $\Downarrow x = \{u \in P : u \ll x\}$ and $w(x) = \{F \in P^{(<\omega)} : F \ll x\}$. Let $K(P) = \{k \in P : k \ll k\}$. Points in $K(P)$ are called *compact elements* of P.

Definition 1.1. Let P be a dcpo.

1. P is called a *continuous domain*, if for each $x \in P$, $\Downarrow x$ is directed and $x = \vee \Downarrow x$.
2. P is called an *algebraic domain*, if for each $x \in P$, $K(P) \cap \downarrow x$ is directed and $x = \vee(K(P) \cap \downarrow x)$.
3. P is called a *quasicontinuous domain*, if for each $x \in P$, $\{\uparrow F : F \in w(x)\}$ is filtered and $\uparrow x = \cap\{\uparrow F : F \in w(x)\}$.
4. P is called a *quasialgebraic domain*, if for each $x \in P$, $\{\uparrow F \in \mathbf{Fin}P : x \in \uparrow F, F \ll F\}$ is filtered and $\uparrow x = \cap\{\uparrow F \in \mathbf{Fin}P : x \in \uparrow F, F \ll F\}$.

Proposition 1.2. *([2, Proposition II-2.11 and Theorem II-2.14]) For a dcpo P, the following two conditions are equivalent:*

1. *P is quasicontinuous.*
2. *For any $x \in P$ and any $U \in \sigma(P)$ with $x \in U$, there is $F \in P^{(<\omega)}$ such that $x \in \mathrm{int}_{\sigma(P)} \uparrow F \subseteq \uparrow F \subseteq U$.*

Proposition 1.3. *([2, Exercise II-3.24]) For a dcpo P, the following conditions are equivalent:*

1. *P is quasialgebraic.*
2. *For any $x \in P$ and any $U \in \sigma(P)$ with $x \in U$, there is $F \in P^{(<\omega)}$ such that $x \in \mathrm{int}_{\sigma(P)} \uparrow F = \uparrow F \subseteq U$.*

Definition 1.4. ([2, Definition III-2.1], [8, Definition 2.2]) A dcpo P is said to be a *meet-continuous domain* if for any $x \in P$ and any $D \in \mathcal{D}(P)$, $x \leq \vee D$ implies $x \in \mathrm{int}_{\sigma(P)}(\downarrow \cap \downarrow D)$.

Proposition 1.5. *([8, Theorem 2.5]) For a dcpo, the following two conditions are equivalent:*

1. *P is continuous.*
2. *P is quasicontinuous and meet-continuous.*

For algebraic domains, we have the following similar result.

Proposition 1.6. *For a dcpo, the following two conditions are equivalent:*

1. *P is algebraic.*
2. *P is quasialgebraic and meet-continuous.*

Rudin's Lemma, given by Mary Rudin in [10], is a useful tool in non-Hausdorff topology and plays a crucial role in domain theory (see [2, 3, 4, 13]). Here will use the following Jung's version of Rudin's Lemma (see [7, Theorem 4.11], [2, Lemma III-3.3], or [5, Corollary 3.5]).

Lemma 1.7. *(Rudin's Lemma) Let P be a poset and $\{F_i : i \in I\} \subseteq P^{(<\omega)}$. If $\{\uparrow F_i : i \in I\}$ is a filtered family, then there is a directed subset D of $\bigcup_{i\in I} F_i$ that meets all F_i.*

As a corollary of Lemma 1.7, we have the following useful result (cf. [5, Corollary 3.9]).

Corollary 1.8. *Let P be a dcpo, U a Scott-open set of P and $\{\uparrow F_i : i \in I\} \subseteq$ **Fin** P be a filtered family with $\bigcap_{i\in I} \uparrow F_i \subseteq U$. Then $\uparrow F_i \subseteq U$ for some $i \in I$.*

Theorem 1.9. *Let P be a quasicontinuous domain that fails to be quasialgebraic. Then there is a surjective monotone Lawson continuous function $f : P \to [0, 1]$.*

Proof. First, we define a binary relation $\sqsubseteq$ on **Fin**P by $\uparrow F \sqsubseteq \uparrow G$ if and only if $\uparrow F \subseteq \mathrm{int}_{\sigma(P)} \uparrow G$. As P is quasicontinous, by Proposition 1.2 the relation $\sqsubseteq$ on **Fin**P is idempotent, that is, $\sqsubseteq$ is transitive and satisfies the following interpolation property:

(INT) For all $\uparrow F, \uparrow G \in$ **Fin**F with $\uparrow F \sqsubseteq \uparrow G$, there exists $\uparrow H \in$ **Fin**P such that $\uparrow F \sqsubseteq \uparrow H \sqsubseteq \uparrow G$.

Second, for any $W \in \sigma(P)$, let $\mathcal{F}_W = \{\uparrow F \in$ **Fin**$P : \uparrow F \in \sigma(P), \uparrow F \subseteq W\} = \{\uparrow F \in$ **Fin**$P : \mathrm{int}_{\sigma(P)} \uparrow F = \uparrow F \subseteq W\}$. As P is not quasialgebraic, by Proposition 1.3 there is $U \in \sigma(P) \backslash \{\emptyset\}$ such that $U \neq \bigcup \mathcal{F}_U$. Choose $x \in U \backslash \bigcup \mathcal{F}_U$. Then by Proposition 1.2 there is $\uparrow F \in$ **Fin**P with $x \in \mathrm{int}_{\sigma(P)} \uparrow F \subseteq \uparrow F \subseteq U$. Then $\uparrow x \notin \mathcal{F}_U, \uparrow F \notin \mathcal{F}_U$ (i.e., $\mathrm{int}_{\sigma(P)} \uparrow x \neq \uparrow x$ and $\mathrm{int}_{\sigma(P)} \uparrow F \neq \uparrow F$), and $\uparrow x \sqsubseteq \uparrow F$. Let $B = \{\frac{m}{2^n} : m, n \in \mathbb{N}, m \leq 2^n\}$ be the set of dyadic rational numbers in $[0, 1]$. Then by the interpolation property of $\sqsubseteq$ and induction, we can get a family $\{\uparrow F(b) : b \in B\} \subseteq$ **Fin**P such that

(i) $\uparrow F(0) = \uparrow F, \uparrow F(1) = \uparrow x$, and
(ii) $\uparrow F(b_2) \sqsubseteq \uparrow F(b_1)$ whenever $b_1 < b_2$.

For any $b \in B$, we have $\uparrow F(1) \sqsubseteq \uparrow F(b) \sqsubseteq \uparrow F(0)$. So $x \in \mathrm{int}_{\sigma(P)} \uparrow F(b) \subseteq \uparrow F(b) \subseteq \mathrm{int}_{\sigma(P)} \uparrow F \subseteq \uparrow F \subseteq U$. As $x \in U \backslash \bigcup \mathcal{F}_U$, we get the following two conclusions:

(iii) $\mathrm{int}_{\sigma(P)} \uparrow F(b) \subsetneq \uparrow F(b)$ for any $b \in B$, and
(iv) $\uparrow F(b_2) \subsetneq \mathrm{int}_{\sigma(P)} \uparrow F(b_1) \subsetneq \uparrow F(b_1)$ whenever $b_1 < b_2$.

Define a function $f : P \to [0,1]$ by

$$f(y) = \begin{cases} \sup\{b \in B : b \in \uparrow F(b)\}, & y \in \uparrow F, \\ 0, & \textit{otherwise.} \end{cases}$$

As $t = \vee(\downarrow t \cap (B \backslash \{t\})$ for all $t \in [0,1]$, using (i) and (ii) we can easily verify that $f(y) = \sup\{b \in B : b \in \mathrm{int}_{\sigma(P)} \uparrow F(b)\}$ for any $y \in \uparrow F$. Clearly, f is monotone, $f(x) = 1$ and $f(P \backslash \uparrow F) = \{0\}$ (note that $P \backslash \uparrow F \neq \emptyset$ by $\uparrow F \notin \sigma(P)$).

Now we prove that f is Lawson continuous. For each $\alpha \in (0,1]$ and each $\beta \in [0,1)$, we have

$$\begin{aligned} f^{-1}([0,\alpha)) &= \{y \in P : f(y) < \alpha\} \\ &= (P \backslash \uparrow F) \cup \{y \in \uparrow F : \sup\{b \in B : b \in \uparrow F(b)\} < \alpha\} \\ &= (P \backslash \uparrow F) \cup \left(\uparrow F \cap \bigcup_{b \in B, b < \alpha} (P \backslash \uparrow F(b))\right) \\ &= (P \backslash \uparrow F) \cup \bigcup_{b \in B, b < \alpha} (P \backslash \uparrow F(b)) \in \omega(p), \text{ and} \end{aligned}$$

$$\begin{aligned} f^{-1}((\beta,1]) &= \{y \in P : f(y) > \beta\} \\ &= \{y \in \uparrow F : \sup\{b \in B : b \in \uparrow F(b)\} < \alpha\} \\ &= \{y \in \uparrow F : \sup\{b \in B : b \in \mathrm{int}_{\sigma(P)} \uparrow F(b)\} > \beta\} \\ &= \bigcup_{b \in B, b > \beta} \mathrm{int}_{\sigma(P)} \uparrow F(b) \in \sigma(P). \end{aligned}$$

Therefore, f is Lawson continuous.

Finally, we show that f is surjective. Let $t \in (0,1)$. For any $b \in B$ with $b < t$. Choose $b' \in B$ with $b < b' < t$. Then for any $r \in B$ with $t < r$, we have $\mathrm{int}_{\sigma(P)} \uparrow F(r) \subsetneq \uparrow F(r) \subsetneq \mathrm{int}_{\sigma(P)} \uparrow F(b') \subsetneq \uparrow F(b') \subsetneq \mathrm{int}_{\sigma(P)} \uparrow F(b) \subsetneq \uparrow F(b)$, and hence $f^{-1}((t,1]) = \bigcup_{r \in B, r > t} \uparrow F(b') \subseteq \mathrm{int}_{\sigma(P)} \uparrow F(b') \subsetneq \uparrow F(b)$. So $f^{-1}((t,1]) \subseteq \bigcap_{b \in B, b < t} \uparrow F(b)$. By (iv) $\{\uparrow F(b) : b \in B, b < t\} \subseteq \mathbf{Fin}P$ is filtered. Then $\bigcap_{b \in B, b < t} \uparrow F(b) \neq f^{-1}((t,1])$, for otherwise $\bigcap_{b \in B, b < t} \uparrow F(b) = f^{-1}((t,1])$ would imply $\uparrow F(b) \subseteq f^{-1}((t,1])$ for some $b \in B$ with $b < t$ by Corollary 1.8, whence $\uparrow F(b) = f^{-1}((t,1]) \in \sigma(P)$, a contradiction. Choose $z \in \bigcap_{b \in B, b < t} \uparrow F(b) \backslash f^{-1}((t,1])$. Then $f(z) = t$. This completes the proof that $f(P) = [0,1]$. $\square$

Remark 1.10. Let X be a topological space and $\mathcal{O}(X)$ be the set of all open subsets of X. Define a relation $\sqsubseteq_t$ on $\mathcal{O}(X)$ by $V \sqsubseteq_t W$ if and only if $\overline{V} \subseteq W$, where $\overline{V}$ is the closure of V in X. Obviously, $\sqsubseteq_t$ is transitive. If X is a normal space, then $\sqsubseteq_t$ satisfies the interpolation property (INT). For a pair A, B of disjoint closed subsets of a normal space X, by the normality of X, there is an open set U such that $A \subseteq U \subseteq \overline{U} \subseteq X \backslash B$. The main technique used in the proof of Urysohn's lemma is to construct a family $\{U(b) : b \in B\} \subseteq \mathcal{O}(X)$ (where $B = \{\frac{m}{2^n} : m, n \in \mathbb{N}, m \leq 2^n\}$) such that

1. $U(1) = U, U(0) = X \backslash B$, and
2. $U(b_2) \sqsubseteq_t U(b_1)$ whenever $b_1 < b_2$,

and define the function $f : X \to [0, 1]$ by

$$f(y) = \begin{cases} \sup\{b \in B : b \in U(b)\}, & y \in X \backslash B, \\ 0, & \textit{otherwise.} \end{cases}$$

Remark 1.11. In [11], we first drew on the technique of proof of Urysohn's lemma to present a direct approach to the construction of fundamental homomorphisms of M-continuous lattices into the unit interval $[0, 1]$ and show that M-continuous lattices admit enough such homomorphisms into $[0, 1]$ to separate points. Later, in [12] it was adopted to constructively prove that if P is a quasicontinuous domain and all lower closed subsets in $(P, \lambda(P))$ are closed in $(P, \omega(P)))$, then $(P, \lambda(P))$ is strictly completely regular ordered space, which gave a partial answer to a problem posed by Jimmie Lawson in [9].

Since $|[0, 1]| = c > \omega$, from Theorem 1.9 we directly deduce the following.

Corollary 1.12. *Every countable quasicontinuous domain is quasialgebraic.*

By Proposition 1.5, Proposition 1.6 and Corollary 1.12, we get the following.

Corollary 1.13. *([6, Corollary 1]) Every countable continuous domain is algebraic.*

In [6], Jia et al. have proved that if P is a continuous domain that fails to be algebraic, then there are two Scott continuous mappings $f : P \to [0, 1]$ and $g : [0, 1] \to p$ such that $f \circ g = id_{[0,1]}$. Then $g([0, 1])$ is a sub dcpo of P satisfying $g([0, 1]) \cong [0, 1]$. So there naturally arise the following two problems.

Problem 1.14. If P is a quasicontinuous domain that fails to be quasialgebraic, does it admit the unit interval $[0, 1]$ as its Scott-continuous retract.

Problem 1.15. If P is a quasicontinuous domain that fails to be quasialgebraic, is there a sub dcpo C of P with $C \cong [0, 1]$?

References

[1] S. Abramsky, A. Jung, Domain theory, in: S. Abramsky, D. Gabbay, and T. Maibaum (eds.), Semantic Structures, volume 3 of Handbook of Logic in Computer Science, Clarendon Press, 1994, pp. 1-168.

[2] G. Gierz, K. Hofmann, K. Keimel, J. Lawson, M. Mislove, D. Scott, Continuous Lattices and Domains, Encycl. Math. Appl., vol. 93, Cambridge University Press, 2003.

[3] G. Gierz, J. Lawson, A. Stralka, Quasicontinuous posets, Houst. J. Math., 9 (1983), 191-208.

[4] J. Goubault-Larrecq, Non-Hausdorff Topology and Domain Theory, New Mathematical Monographs, vol. 22, Cambridge University Press, 2013.

[5] R. Heckmann, K. Keimel, Quasicontinuous domains and the Smyth powerdomain, Electron. Notes Theor. Comput. Sci., 298 (2013), 215-232.

[6] X. Jia, Q. Li, W. Luan, Separating continuous domains from algebraic domains, Bull. Lond. Math. Soc., 2025, doi:10.1112./blms.70227.

[7] A. Jung, Cartesian Closed Categories of Domains, CWI Tracts vol. 66, Centrum voor Wiskunde en Informatica, Amsterdam, 1989.

[8] H. Kou, Y. Liu, M. Luo, On meet-continuous dcpos, in: Domain Theory, Logic and Computation, volume 3 of Semantic Structures in Computation, Springer Netherlands, 2003, pp. 117-135.

[9] J. Lawson, Order and strongly sober compactifications, in: Topology and Category Theory in Computer Science, Oxford Press, 1991, pp. 179-205.

[10] M. Rudin, Directed sets which converge, in: General Topology and Modern Analysis, University of California, Riverside, 1980, pp. 305-307. Academic Press, 1981.

[11] X. Xu, Construction of homomorphisms of M-continuous lattices, Trans. of Amer. Math. Soc., 347(1995), 3167-3175.

[12] X. Xu, The Lawson topology on quasicontinuous domains, in: Domains and Processes, volume 1 of Semantic Structures in Computation, Kluwer Academic Publishers, 2001, pp. 33-40.

[13] X. Xu, Order and Topology, Science Press, Beijing, 2022.