OPEN PROBLEMS IN COMPUTABILITY THEORY AND DESCRIPTIVE SET THEORY

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Abstract. These open problems were presented in the Problem Sessions held during the Tianyuan Workshop on Computability Theory and Descriptive Set Theory, June 22-26, 2026. The problems are organized into sections named after their contributors, in the order of their presentations during the workshop. Notes were taken and compiled by Wei Dai, Xiangxi Hu, Yingying Jiang, Ruiwen Li, Tianhao Wang, Xu Wang, Jie Zou.

Contents

1. George Barmpalias 2
2. Andre Nies 3
3. Takayuki Kihara 5
4. Liang Yu 9
5. Jialiang He 9
6. Chieu-Minh Tran 11
7. Theodore Slaman 12
8. Hang Zhang 12
9. Daniel Turetsky 13
10. Philip Welch 13
11. David Schrittesser 14
12. Su Gao 14

Date: June 2026.

1. George Barmpalias

The following problems concern Turing ideals generated by sufficiently random reals.

Problem 1. (Question 1, Levin) Are the following true?

(1) Can every chain of random reals in the Turing degrees be combined into a single random real?

(2) Can every Turing ideal generated by random reals be generated by the columns of a single random real?

Notation. If $\alpha \in 2^\omega$, let $\alpha^{(i)}$ denote the real obtained from α by deleting the digits $\alpha(k)$ at positions k that are divisible by 2^i.

(1) A *chain* is a sequence $(x_i)_{i\in\mathbb{N}}$ of reals such that

$$x_i <_T x_{i+1}$$

for every $i \in \mathbb{N}$.

(2) If B is a chain of reals, the Turing ideal generated by B is

$$[B] := \{z : \exists x \in B,\ z \leq_T x\}.$$

In particular, for a chain $(x_i)_{i\in\mathbb{N}}$,

$$[(x_i)] := \{z : \exists i,\ z \leq_T x_i\}.$$

(3) For each real α, define

$$[\alpha] := \{z : \exists i,\ z \leq_T \alpha^{(i)}\}.$$

(4) The notation $z \approx x$ means that z and x differ on finitely many bits.

(5) The real $\bigoplus_i x_i$ is the infinite join of all the reals x_i.

Proposition (Barmpalias and Zhang). Let $(x_i)_{i\in\mathbb{N}}$ be a chain of random reals such that x_{i+1} is random relative to x_i for every i. Then there are reals $z_i \approx x_i$ such that $\bigoplus_i z_i$ is random. The same conclusion holds for all standard randomness notions.

This proposition gives a partial positive answer to Question 1(1) under the additional assumption of relative randomness. Question 1(2) asks for the stronger conclusion that the ideal generated by a random chain is exactly of the form $[\alpha]$ for some random real α.

Problem 2. (Question 1a, Levin) Is it true that, for every $X \subseteq 2^\omega$ with $\mu(X) = 1$, there exists $Y \subseteq 2^\omega$ with $\mu(Y) = 1$ such that, for every chain $B \subseteq Y$, there is some $\alpha \in X$ satisfying $[B] = [\alpha]$?

The class $\mathrm{ML}(\emptyset')$ of $\emptyset'$-random reals, equivalently the class of 2-random reals, is a special case.

Problem 3. (Question 1b) Is there a chain $B \subseteq \mathrm{ML}(\emptyset')$ such that $[B] \neq [\alpha]$ for every $\alpha \in \mathrm{ML}(\emptyset')$?

There are two obstacles to constructing such a chain. If α is 2-random, then the ideal $[\alpha]$ has the following properties:

(1) For every $x \in [\alpha]$, the ideal $[\alpha]$ contains an x'-random real.

(2) The jumps of the columns $\alpha^{(i)}$ form a chain and, in particular, are pairwise distinct.

Problem 4. (Question 1c) Toward a negative answer to Levin's question, are the following true?

(1) Are there 2-random reals $x <_T z$ such that
$$x' \equiv_T z'?$$

(2) Are there 2-random reals $x <_T z$ such that
$$|K(x \upharpoonright n) - K(z \upharpoonright n)| = O(1)?$$

(3) Is there a Turing ideal B generated by a chain of 2-random reals such that B contains no x-random real for any $x \in B$?

(4) What are the jumps of 2-random reals? See Barmpalias, Downey, and Ng (2008).

Problem 5. (Question 2) Is there an injective one-way function on the reals? More precisely, is there a partial computable real function f such that

(1) f is injective and preserves randomness;

(2) the domain of f has positive Lebesgue measure;

(3) $f(x) <_T x$ for a positive-measure set of reals x in the domain of f?

Such a function can be used to produce a chain of 2-random reals with the same jump in the Turing degrees.

2. Andre Nies

Let M be a countably infinite ω-categorical structure in a finite first-order language, and let
$$G = \mathrm{Aut}(M),$$

equipped with the topology of pointwise convergence. Let $\mathrm{Aut}(G)$ denote the group of continuous automorphisms of the topological group G, and let

$$\mathrm{Out}(G) := \mathrm{Aut}(G)/\,\mathrm{Inn}(G)$$

be its outer automorphism group.

Problem 6. Let M be an ω-categorical structure in a first order finite language. Is $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ a finite group?

For some classical examples like $M = (\mathbb{Q}, <)$ [2], the statement of the problem is true.

Problem 7. Does $\leq_{SJT}$ imply $\leq_{LR}$?

Assume throughout that $A, B \subset \mathbb{N}$. Then

$$A \leq_{SJT} B$$

iff for every computable order function h there exists a uniformly B-c.e. trace (T_n) satisfying that

$$|T_n| \leq h(n)$$

and

$$J^A(n) \downarrow \Longrightarrow J^A(n) \in T_n.$$

Also we have the definition of $\leq_{LR}$.

$$A \leq_{LR} B \iff \mathrm{MLR}^B \subseteq \mathrm{MLR}^A.$$

Let SJT be the class of $A \subset \mathbb{N}$ such that $A \leq_{SJT} \emptyset$, then SJT $\subsetneq$ K-trivial. This inspires our problem. In [1], the authors have proved that, for A, B K-trivial sets,

$$A \leq_{SJT} B \iff \forall Y\ \omega\text{-c.a. and } Y \in \mathrm{MLR}\Big(A \leq_T B \oplus Y\Big).$$

For more details one can read [1].

Problem 8. Is there a Π^0_1 class C such that $|C \cap \mathrm{Rec}| = \infty$ and $C \cap \mathrm{Rec}$ is MAD in recursive sets?

Here the ”MAD” means maximal almost disjoint.

References

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3. Takayuki Kihara

The following problems concern a realizability-theoretic refinement of many-one and Wadge reducibility.

For sets $A, B \subseteq \omega^\omega$, define
$$A \leq_m B$$
if there is a computable function
$$\theta \colon \omega^\omega \to \omega^\omega$$
such that
$$x \in A \quad \Longleftrightarrow \quad \theta(x) \in B$$
for every $x \in \omega^\omega$. Replacing "computable" by "continuous" gives Wadge reducibility, denoted by
$$A \leq_W B.$$

For formulas, the truth-preserving map on instances is not sufficient. The witnesses for the two formulas must also be uniformly convertible.

Let $\alpha \Vdash \varphi(x)$ mean that α is a witness, or realizer, for the truth of $\varphi(x)$. The witness relation is defined inductively from the logical structure of φ; see [2, Definition 2.10].

A formula φ is realizability-theoretically many-one reducible to a formula ψ, written
$$\varphi \leq_m \psi,$$
if there are computable functions η, r^-, r^+ such that

(1)
$$\varphi(x) \quad \Longleftrightarrow \quad \psi(\eta(x));$$

(2) If $\alpha \Vdash \varphi(x)$, then
$$r^-(\alpha, x) \Vdash \psi(\eta(x));$$

(3) If $\beta \Vdash \psi(\eta(x))$, then
$$r^+(\beta, x) \Vdash \varphi(x).$$

This is also called *Levin reducibility*; see [2, Definition 2.13]. Its continuous analogue gives the corresponding realizability-theoretic version of Wadge reducibility.

Existentially quantified variables may depend on all preceding universally quantified variables. For example, a witness for

$$\forall a\,\exists b\,\forall c\,\exists d\,\forall e\,\exists f\,\forall g\,\varphi(a,b,c,d,e,f,g,x)$$

consists of three functions

$$F_0\colon \omega\to\omega, \qquad F_1\colon \omega^2\to\omega, \qquad F_2\colon \omega^3\to\omega$$

such that

$$\forall a\,\forall c\,\forall e\,\forall g\,\varphi(a,F_0(a),c,F_1(a,c),e,F_2(a,c,e),g,x).$$

Thus, $F_0(a)$ witnesses b, $F_1(a,c)$ witnesses d, and $F_2(a,c,e)$ witnesses f.

In addition to the ordinary quantifiers $\exists$ and $\forall$, define

$$\exists^\infty n\,\varphi(n) \quad\Longleftrightarrow\quad \forall m\,\exists n\geq m\,\varphi(n),$$

and

$$\forall^\infty n\,\varphi(n) \quad\Longleftrightarrow\quad \exists m\,\forall n\geq m\,\varphi(n).$$

A finite sequence

$$\overline{Q}\in\{\exists,\forall,\exists^\infty,\forall^\infty\}^{<\omega}$$

is called a *quantifier pattern*. If

$$\overline{Q}=Q_0Q_1\cdots Q_\ell,$$

then a formula of the form

$$Q_0n_0\,Q_1n_1\cdots Q_\ell n_\ell\,\theta(n_0,\dots,n_\ell,x),$$

where θ is bounded, is called an $\overline{Q}$-formula.

For each quantifier pattern $\overline{Q}$, let

$$\langle\overline{Q}\rangle$$

denote a fixed $\overline{Q}$-complete formula.

Consider

$$\mathrm{Bdd}:=\{x\in\omega^\omega : \exists n\,\forall k\,x(k)\leq n\},$$

and

$$\mathrm{FIN}:=\{x\in\omega^\omega : \exists n\,\forall m\geq n\,x(m)=0\}.$$

Although both are classically Σ^0_2 properties, their witness structures are different. The first has the quantifier pattern $\exists\forall$, whereas the second has the pattern $\forall^\infty$.

The realizability-theoretic classification of Σ^0_2 quantifier patterns consists of exactly three degrees:

$$\langle\forall^\infty\rangle <_m \langle\forall^\infty\forall\rangle <_m \langle\exists\forall\rangle.$$

The abstract treatment of this classification is given in [1].

The more concrete analysis of the Σ^0_3 and Π^0_3 levels is given in [2]. The known numbers of equivalence classes are as follows:

(1) There is exactly one many-one equivalence class of Π^0_2 quantifier patterns, represented by

$$\forall\exists.$$

(2) There are exactly three many-one equivalence classes of Σ^0_2 quantifier patterns, represented by

$$\exists\forall, \qquad \forall^\infty\forall, \qquad \forall^\infty.$$

(3) There are exactly three many-one equivalence classes of Σ^0_3 quantifier patterns, represented by

$$\exists\forall\exists, \qquad \forall^\infty\exists^\infty, \qquad \forall^\infty\exists.$$

(4) There are exactly five many-one equivalence classes of Π^0_3 quantifier patterns, represented by

$$\forall\exists\forall, \qquad \exists^\infty\forall^\infty\forall, \qquad \exists^\infty\forall, \qquad \forall\forall^\infty\forall, \qquad \forall\forall^\infty.$$

The corresponding classifications at the fourth level are still in progress.

Problem 9. Determine the equivalence classes of quantifier patterns at the fourth level of the arithmetical hierarchy:

(1) How many $\equiv_m$-equivalence classes of Σ^0_4 quantifier patterns are there?

(2) How many $\equiv_m$-equivalence classes of Π^0_4 quantifier patterns are there?

(3) How many $\equiv_{dm}$-equivalence classes of Σ^0_4 quantifier patterns are there?

Problem 10. Is there a decision procedure which, given quantifier patterns

$$\overline{P}, \overline{Q} \in \{\exists, \forall, \exists^\infty, \forall^\infty\}^{<\omega},$$

decides whether

$$\langle\overline{P}\rangle \leq_m \langle\overline{Q}\rangle?$$

Let $x \in \omega^\omega$ code a countable partial order

$$P_x = (\omega, \leq_{P_x}).$$

Consider the following properties:

(1) IFPO(x) asserts that P_x is ill-founded, that is, there is a function $f \in \omega^\omega$ such that

$$f(n+1) <_{P_x} f(n)$$

for every n.

(2) Anti(x) asserts that P_x contains an infinite antichain.

(3) NWPO(x) asserts that P_x is not a well-partial-order.

(4) BadPO(x) asserts that there is a bad sequence in P_x:

$$\exists f \in \omega^\omega\, \forall i < j\, (f(i) \not\leq_{P_x} f(j)).$$

On the blackboard, the last condition was abbreviated by

$$\exists f \in \omega^\omega\, \forall n\, (f(n) \not\leq_{P_x} f(n+1)).$$

Classically, the failure of the well-partial-order property can be described using descending sequences, antichains, or bad sequences. From the realizability-theoretic viewpoint, these formulas carry different types of witnesses and therefore need not have the same many-one degree.

The blackboard records

$$\mathrm{IFPO} \equiv_m \mathrm{Anti},$$

and asks for the precise relationship between these problems and BadPO.

For $n \geq 2$, let IFPO_n and BadPO_n denote the corresponding restrictions to partial orders of width at most n. For width one, both problems reduce to the ill-founded linear-order problem, denoted by IFLO.

Problem 11. Determine the realizability-theoretic many-one and Wadge relationships among the partial-order problems above. In particular:

(1) Is

$$\mathrm{IFPO} \equiv_m \mathrm{BadPO}?$$

(2) For which $m, n \geq 2$ does

$$\mathrm{BadPO}_n \leq_m \mathrm{IFPO}_m$$

hold?

(3) For which $m, n \geq 2$ does

$$\mathrm{IFPO}_n \not\leq_m \mathrm{BadPO}_m$$

hold?

References

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[2] T. Kihara, *The arithmetical hierarchy: a realizability-theoretic perspective.* arXiv:2410.15795 (2024).

4. Liang Yu

A set $A \subseteq \mathbb{R}^2$ is called a *Kakeya set* if, for every $\alpha \in \mathbb{R}$, there exists $C_\alpha \in \mathbb{R}$ such that

$$y = \alpha x + C_\alpha \quad \Longrightarrow \quad (x, y) \in A$$

for every $(x, y) \in \mathbb{R}^2$. Equivalently,

$$\{(x, y) \in \mathbb{R}^2 : y = \alpha x + C_\alpha\} \subseteq A$$

for every $\alpha \in \mathbb{R}$.

Problem 12. Is there a Kakeya set $A \subseteq \mathbb{R}^2$ such that

$$x \oplus y \geq_T \emptyset'$$

for every $(x, y) \in A$?

A Kakeya set A is called *minimal* if every proper subset $B \subsetneq A$ is not a Kakeya set.

Problem 13. Does every Kakeya set A contain a minimal Kakeya set $B \subseteq A$?

5. Jialiang He

These notes concern three related definability questions in infinite combinatorics and Borel model theory. Throughout, we identify each subset $A \subseteq \omega$ with its characteristic function $\chi_A \in 2^\omega$. Under this coding, $[\omega]^\omega = \{A \subseteq \omega : \text{A is infinite}\}$ is a G_δ, hence Borel, subspace of 2^ω.

Definition 14. A set $\mathcal{A} \subseteq [\omega]^\omega$ is called an *independent family* iff for any pairwise distinct

$$A_1, \dots, A_n, B_1, \dots, B_m \in \mathcal{A},$$

we have

$$|A_1 \cap \cdots \cap A_n \cap B_1^c \cap \cdots \cap B_m^c| = \aleph_0.$$

An independent family $\mathcal{A}$ is called a *maximal independent family*, if it is maximal with respect to inclusion among independent families.

There is a basic negative theorem is due to Miller.

Theorem 15 (Miller [2])**.** There is no $\mathbf{\Sigma}^1_1$ maximal independent family.

Definition 16. A family $\mathcal{F} \subseteq \omega^\omega$ is called an *independent family of functions* if for every $m \geq 1$, for all pairwise distinct

$$f_0, \dots, f_{m-1} \in \mathcal{F},$$

and for all values

$$j_0, \dots, j_{m-1} \in \omega,$$

the set

$$\{n \in \omega : f_0(n) = j_0,\ f_1(n) = j_1,\ \dots,\ f_{m-1}(n) = j_{m-1}\}$$

is infinite.

An independent family of functions $\mathcal{F} \subseteq \omega^\omega$ is called *maximal* if there is no $g \in \omega^\omega \setminus \mathcal{F}$ such that $\mathcal{F} \cup \{g\}$ is still an independent family of functions.

Problem 17. Is there a $\mathbf{\Sigma}^1_1$ maximal independent family of functions?

Definition 18. A set $X \subseteq \mathbb{R}^2$ is called a *two-point set*, if for every line $L \subseteq \mathbb{R}^2$,

$$|X \cap L| = 2.$$

Theorem 19 (Larman [1])**.** There is no F_σ two-point set.

Since every F_σ set is Borel, Larman's theorem rules out the lowest nontrivial possibility in the Borel hierarchy, but it does not settle the following problem.

Problem 20. Is there a Borel two-point set?

Miller showed that, assuming Gödel's axiom of constructibility $V = L$, there is a $\mathbf{\Pi}^1_1$, i.e., coanalytic two-point set [2].

Definition 21. A *standard Borel space* is a measurable space isomorphic to the Borel space of a Polish space. A *Borel field* is a field $(F, +, \times)$ whose universe is a standard Borel space and whose field operations $+, \times$ are Borel.

Definition 22. Let F be a Borel field. A *Borel algebraically closed extension* of F is a Borel field K, together with a Borel injective field homomorphism

$$\iota : F \hookrightarrow K,$$

such that K is algebraically closed.

Problem 23 (Montalbán–Nies [3])**.** Does every Borel field have a Borel algebraically closed extension?

References

[1] D. G. Larman, *A problem of incidence*, Journal of the London Mathematical Society **43** (1968), 407–409.

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6. Chieu-Minh Tran

It is about Morley's categoricity in power theorem. According to [1], there is the following consequence.

Theorem 24. For a countable theory T, if it is κ categorial for some $\kappa > \aleph_0$, then it is λ categorial for all uncountable λ.

And [2] replaced κ in the above theorem by recursion theoretic complexity (arithmetic degree). Then, the sequel goes to categoricity without Choice. The first research about this topic date back to [3] shows that we still have the theorem holds if we assume ZF + there is some uncountable well ordered set of reals.

Theorem 25. [ZF + there is an uncountable well ordered set of reals].

The following conditions on a countable (first order) T are equivalent:

(1) T is categorical in some cardinal $\aleph_\alpha > \aleph_0$,in $\mathbf{V}$, of course

(2) T is categorical in every cardinal $\aleph_\beta > \aleph_0$, in $\mathbf{V}$, of course

(3) T is, in $\mathbf{L}[T]$, totally transcendental (i.e. $\aleph_0$-stable) with no two cardinal models (i.e., for no model M of T and formula $\varphi(x, y) \in \mathbf{L}(\tau_T)$ and $\bar{a} \in {}^{\ell g(\bar{y})}M$ do we have $\aleph_0 \leq |\varphi(M, \bar{a})| < \|M\|$ and $\|M\|$ is a cardinal, i.e. the set of elements of M is well-orderable hence its power is a cardinal)

(4) If $\mathbf{V}' \subseteq \mathbf{V}$ is a transitive class extending $\mathbf{L}$, $T \in \mathbf{V}'$ and $\mathbf{V}'$ satisfies ZFC, then the conditions in (3) hold

(5) For some $\mathbf{V}'$ clause (4) holds.

However, for non-well-ordered cardinals, the situation deteriorates quickly. The sequel is the following. For such a theory T, we want to show it is very close to a stongly minimal T'. It is like a vector space over fixed field or algebraically closed field. And since there is no well order, it is hard to take a basis and match them to one another. So there is the question.

Problem 26. For vector space over fixed $\mathbb{K}$, is choose a basis require the full Axiom of Choice?

References

[1] M. Morley, Categoricity in power, *Transactions of American Mathematical Society*, 114 (1965), 514-538,

[2] J.L Goh, C.-M. Tran, Categoricity without Power, 2026, arXiv:2605.00697

[3] S.Shelah Model theory without choice? Categoricity, *Journal of Symbolic Logic*, 74 (2) (2009): 361-401,

7. Theodore Slaman

In 1954, [1] shows that for $E \subseteq \mathbb{R}^2$ an analytic set with Hausdorff dimension 1. Then, for almost all θ, the projection of E onto a line with slope θ has Hausdorff dimension 1. Later in 1969, [2] shows that CH could imply there is some $E \subseteq \mathbb{R}^2$ with Hausdorff dimension 1 and for all θ, projection of E onto a line of slope θ has Hausdorff dimension 0.

Problem 27. Is there a Davies example probably in ZFC?

References

[1] J.M.Marstrand, The dimension of Cartesian product sets, *Mathematical Proceedings of the Cambridge Philosophical Society*, 1954, 50(2), 198-202.
[2] R.O.Davies, Measures of Hausdorff Type. *Journal of the London Mathematical Society*, 1969, s2-1, 30-34.

8. Hang Zhang

Moticated by "$hyper - hyperfinite \implies hpperfinite$"?

Example 28. (Calderoni & Sinapova) Let M be a countable model for large fragment ZFC. Suppose $M \models \kappa$ is measurable (witnessed by $\mathcal{U}$). If $M \models \mathbb{P} = \{(s, A) : s = \langle s_0, \ldots, s_n \rangle,\ n \in \omega,\ A \in \mathcal{U}\}$, with order $(s, A) \leq (t, B) \iff t \subseteq s \wedge s \setminus t \subseteq B \wedge A \subseteq B$. Then for any generic sequences of $\mathbb{P}$ over M,

$$(*) \qquad M[\vec{\alpha}] = M[\vec{\beta}] \iff \exists N \forall n > N,\ \alpha(n) = \beta(n).$$

Problem 29. Find more M, $\mathbb{P} \in M$ such that $(*)$ is true.

Remark 30. If $\mathbb{P} = \mathbb{C}$, then "$x \sim y \iff M[x] = M[y]$" is hyper-hyperfinite.

Problem 31. (JKL and Smythe)Is "$\sim$" hyperfinite?

Problem 32. Suppose there is M, $\mathbb{P} \in M$ such that $(*)$ is true, can we deduce that $M \models [\exists\text{large cardinal}]$?

Remark 33. If

$$\langle 0, \cdots, 0, 1, 0, \cdots, 0, 1, 1, 0 \cdots \rangle$$

is Cohen, then

$$\langle 1, \cdots, 1, 0, 1, \cdots, 1, 0, 0, 1 \cdots \rangle$$

is also Cohen.

References

[1] F. Calderoni, D. Sinapova, *Forcing, genericity and countable Borel equivalence relations*, preprint, 2025, arXiv: 2503.14811.

9. Daniel Turetsky

If $\mathbf{a} < \mathbf{b} < \mathbf{c}$ are Turing degrees, and $\mathbf{a}$, $\mathbf{c}$ contain different Martin-Löf randoms, then must $\mathbf{b}$ contain one? The answer is no.

Problem 34. Does this hold with a larger degree notion, for example, LR-degrees?

10. Philip Welch

First we recall some basic notions in [1]. Let β be an ordinal. Suppose M is a nonstandard model of KP, and $\{\zeta_i : i \in \mathbb{N}\}$, $\{s_i : i \in \mathbb{N}\}$ are sequences of M-ordinals. If $(M, \{\zeta_i : i \in \mathbb{N}\}, \{s_i : i \in \mathbb{N}\})$ satisfies the following:

(1) M extends L_β;

(2) for all i, $\zeta_i < \zeta_{i+1} < \beta$ and $s_{i+1} <^M s_i$;

(3) for all i, $L_{\zeta_i} \prec_{\Sigma_{m+1}} L^M_{s_i}$;

(4) for all i, $L^M_{s_{i+1}} \prec_{\Sigma_{m-1}} L^M_{s_i}$,

then we call this triple a strong Σ_{m+1}-nesting on β. And we say β supports a strong Σ_{m+1}-nesting if there exist such nonstandard model M of KP and sequences $\{\zeta_i : i \in \mathbb{N}\}, \{s_i : i \in \mathbb{N}\}$ of M-ordinals for L_β.

We call a function $\Phi : \mathbb{R} \to \mathbb{R}$ an operator ($\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ are identified). An operator Φ is said to be

(1) $\mathbf{\Pi}^1_m$ if there exists a $\mathbf{\Pi}^1_m$ formula ϕ and a parameter Y such that
$$\forall n\, \forall X\, (n \in \Phi(X) \leftrightarrow \phi(n, X, Y))\,;$$

(2) monotone if $X \subseteq Y$ implies $\Phi(X) \subseteq \Phi(Y)$.

The axiom $\mathbf{\Pi}^1_{m+1}{-}\mathsf{MI}$ asserts that for every monotone $\mathbf{\Pi}^1_{m+1}$ operator Φ, there is some prewellordering W such that for all $a \in \text{field}(W) = \text{dom}(W) \cup \text{ran}(W)$, we have
$$W_a = \Phi(W_{<a}).$$
Here
$$W_a = \{b : a <_W b\},$$
$$W_{<a} = \bigcup \{W_b : b <_W a\}.$$
Note that an operator Φ can be iterated by setting $\Phi^0 = \emptyset, \Phi^{<\alpha} = \bigcup_{\gamma<\alpha} \Phi^\gamma$ and $\Phi^\alpha = \Phi(\Phi^{<\alpha})$. Put $\Phi^\infty = \bigcup_\alpha \Phi^\alpha$. We can see that the field of W is Φ^∞.

Problem 35. If $L_\beta \models V = HC$, then
$$\Big(\mathcal{P}(\omega) \cap L_\beta \models \mathbf{\Pi}^1_{m+1}{-}\mathsf{MI}\Big) \iff \beta \text{ supports a strong } \Sigma_{m+1}\text{-nesting.}$$

References

[1] J. P. Aguilera, P. D. Welch, Determinacy on the edge of second-order arithmetic, I, *Journal of the London Mathematical Society*, 2026, 113(3), e70456.

11. David Schrittesser

Let's first recall two classes of groups. They are both subgroups of S_∞ and share similar flavor.

Definition 36. We say a subgroup $G \leq S_\infty$ is *cofinitary* if

$$\forall g \in G \setminus \{1\},\ \mathrm{fix}(g) = \{n \in \omega |\ g(n) = n\}$$

is finite.
We say a group $G \leq S_\infty$ is *maximal cofinitary* if for every proper extension $G \lneq G' \leq S_\infty$, G' is not cofinitary.

Definition 37. We say a subgroup $G \leq S_\infty$ is *finitely periodic* if $\forall g \in G \setminus \{1\}$, $\langle g \rangle_{S_\infty}$ only has finitely many finite orbits.
Similarly, we say a group $G \leq S_\infty$ is *maximal finitely periodic* if for every proper extension $G \lneq G' \leq S_\infty$, G' is not finitely periodic.

Maximal cofinitary groups have been studied by Bart Kastermans in [1]. It's known that maximal cofinitary groups cannot be K_σ, abelian, and there is a coanalytic maximal cofinitary group if assuming the axiom of constructibility. So we can ask similar questions for maximal finitely periodic groups:

Problem 38. Is there a countable or K_σ or just Borel maximal finitely periodic group?

References

[1] B. Kastermans, *The Complexity of Maximal Cofinitary Groups*, Proceedings of the American Mathematical Society, 137 (1) (2009), 307–316.

12. Su Gao

This question is about equivalence relations given by complete first-order theories. Consider a complete first order theory T in a countable language. Let $\mathrm{Mod}(T)$ be the Polish space of countable models of T with universe ω. Let $\cong\restriction_{\mathrm{Mod}(T)}$ denote the isomorphism relation on $\mathrm{Mod}(T)$. Recall two basic equivalence relations in descriptive set theory:

(1) $=$ is the identity relation on $\mathbb{R}$;

(2) $=^+$ is the equivalence relation on $\mathbb{R}^\omega$ that $(x_n)_{n\in\omega} =^+ (y_n)_{n\in\omega}$ if they represent the same countable subset of reals.

Also note the following fact:

Fact 39. There is a complete theory T such that $=$ is Borel bi-reducible to $\cong\restriction_{\mathrm{Mod}(T)}$. For example, T can be chosen as the theory of the equivalence relation on ω with infinitely many equivalence classes of any finite size.
There is a complete theory T' such that $=^+$ is Borel bi-reducible to $\cong\restriction_{\mathrm{Mod}(T')}$. For example, T' can be chosen as the theory of infinitely many independent equivalence relations on ω and every equivalence relation has infinitely many equivalence classes of any finite size.

So we have the following natural question:

Problem 40. Does there exist a complete first-order theory T in a countable language such that $=<_B\cong\restriction_{\mathrm{Mod}(T)}<_B=^+$?

Remark 41. Note that $=<_B=^+$, and there are equivalence relations strictly between them. For example, $E_0, E_\infty, E_0^\omega$. So, they are natural candidates for the complexity of $\cong\restriction_{\mathrm{Mod}(T)}$ for some T if the problem has a positive answer.