

OPEN PROBLEMS IN MATHEMATICAL LOGIC

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ABSTRACT. These open problems were presented in the Problem Sessions held during the Tianyuan Workshop on Definability and Computation, June 22-26, 2026. The problems are organized into sections named after their contributors, in the order of their presentations during the workshop. Notes were taken and compiled by Wei Dai, Xiangxi Hu, Yingying Jiang, Ruiwen Li, Tianhao Wang, Xu Wang, and Jie Zou.

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1. GEORGE BARMPALIAS

The following problems concern Turing ideals generated by sufficiently random reals.

Problem 1.1 (Levin, private communications). Are the following true?

- (1) Can every chain of random reals in the Turing degrees be combined into a single random real?
- (2) Can every Turing ideal generated by random reals be generated by the columns of a single random real?

Notation 1.1. If $\alpha \in 2^\omega$, let $\alpha^{(i)}$ denote the real obtained from α by deleting the digits $\alpha(k)$ at positions k that are divisible by 2^i .

- A *chain* is a sequence $(x_i)_{i \in \mathbb{N}}$ of reals such that

$$x_i <_T x_{i+1}$$

for every $i \in \mathbb{N}$.

- If B is a chain of reals, the Turing ideal generated by B is

$$[B] := \{z : \exists x \in B, z \leq_T x\}.$$

In particular, for a chain $(x_i)_{i \in \mathbb{N}}$,

$$[(x_i)] := \{z : \exists i, z \leq_T x_i\}.$$

- For each real α , define

$$[\alpha] := \{z : \exists i, z \leq_T \alpha^{(i)}\}.$$

- The notation $z \approx x$ means that z and x differ on finitely many bits.
- The real $\bigoplus_i x_i$ is the infinite join of all the reals x_i .

Proposition 1.2 (Barmpalias–Zhang, unpublished). *If $(x_i)_{i \in \mathbb{N}}$ is a sequence of reals and $\bigoplus_{i \leq n} x_i$ is random for each n , then there are reals $z_i \approx x_i$ so that $\bigoplus_i z_i$ is random. The same conclusion holds for k -randomness and weak k -randomness for each $k \in \mathbb{N}$.*

This proposition gives a partial positive answer to Question 1(1) under the additional assumption of relative randomness. Question 1(2) asks for the stronger conclusion that the ideal generated by a random chain is exactly of the form $[\alpha]$ for some random real α .

Problem 1.2 (Levin). Is it true that, for every $X \subseteq 2^\omega$ with $\mu(X) = 1$, there exists $Y \subseteq 2^\omega$ with $\mu(Y) = 1$ such that, for every chain $B \subseteq Y$, there is some $\alpha \in X$ satisfying $[B] = [\alpha]$?

The class $\text{ML}(\emptyset')$ of $\emptyset'$ -random reals, equivalently the class of 2-random reals, is a special case.

Problem 1.3. Is there a chain $B \subseteq \text{ML}(\emptyset')$ such that $[B] \neq [\alpha]$ for every $\alpha \in \text{ML}(\emptyset')$?

There are two obstacles to constructing such a chain. If α is 2-random, then the ideal $[\alpha]$ has the following properties:

- (i) For every $x \in [\alpha]$, the ideal $[\alpha]$ contains an x' -random real.
- (ii) The jumps of the columns $\alpha^{(i)}$ form a chain and, in particular, are pairwise distinct.

Problem 1.4. Toward a negative answer to Levin's Problem 1.2, are the following true?

- (1) Are there 2-random reals $x <_T z$ such that

$$x' \equiv_T z'?$$

- (2) Are there 2-random reals $x <_T z$ such that

$$|K(x \upharpoonright n) - K(z \upharpoonright n)| = O(1)?$$

- (3) Is there a Turing ideal B generated by a chain of 2-random reals such that B contains no x -random real for any $x \in B$?
- (4) What are the jumps of 2-random reals? See [1].

Problem 1.5. Is there an injective one-way function on the reals? More precisely, is there a partial computable real function f such that

- (i) f is injective and preserves randomness,
- (ii) the domain of f has positive Lebesgue measure, and
- (iii) $f(x) <_T x$ for a positive-measure set of reals x in the domain of f ?

Such a function can be used to produce a chain of 2-random reals with the same jump in the Turing degrees.

References

- [1] G. Barmpalias, R. Downey, and K. M. Ng, *Jump inversions inside effectively closed sets and applications to randomness*, The Journal of Symbolic Logic 76 (2011), no. 2, 491–518.

2. ANDRE NIES

Oligomorphic groups. Let M be a countably infinite ω -categorical structure in a finite first-order signature, and let

$$G = \text{Aut}(M),$$

equipped with the topology of pointwise convergence. Let $\text{Aut}(G)$ denote the group of continuous automorphisms of the topological group G , and let

$$\text{Out}(G) := \text{Aut}(G)/\text{Inn}(G)$$

be its outer automorphism group.

Problem 2.1. Let M be an ω -categorical structure in a first order finite language. Is the group $\text{Out}(G)$ finite?

For background see [2], where it is shown that this group is countable. For some classical examples like $M = (\mathbb{Q}, <)$ [3], the statement of the problem is true; in this case $\text{Out}(G)$ has two elements, given by the identity and conjugation by $x \mapsto -x$.

Computability Theory. Assume throughout that $A, B \subset \mathbb{N}$. One writes

$$A \leq_{SJT} B$$

if for every order function h there exists a uniformly B -c.e. trace (T_n) satisfying that

$$|T_n| \leq h(n)$$

and

$$J^A(n) \downarrow \implies J^A(n) \in T_n.$$

Also recall that

$$A \leq_{LR} B \iff \text{MLR}^B \subseteq \text{MLR}^A.$$

Problem 2.2. Does $\leq_{SJT}$ imply $\leq_{LR}$?

We note that $A \leq_{SJT} \emptyset$ implies $A \leq_{LR} \emptyset$, namely, every strongly jump traceable set is K -trivial, equivalently, low for ML-random. For background and references, see [1]. The authors there give several characterisations of $\leq_{SJT}$ on the K -trivials; for instance, for A, B K -trivial,

$$A \leq_{SJT} B \iff \forall Y \text{ } \omega\text{-c.a. and } Y \in \text{MLR} \left(A \leq_T B \oplus Y \right).$$

Let Rec denote the class of recursive sets.

Problem 2.3. Is there a Π_1^0 class C such that $|C \cap \text{Rec}| = \infty$ and $C \cap \text{Rec}$ is maximally almost disjoint among the recursive sets?

References

- [1] N. Greenberg, A. Nies and D. Turetsky. *Characterising SJT reducibility*, The Journal of Symbolic Logic (2026): 1–23.
- [2] A. Nies and G. Paolini. *Oligomorphic groups, their automorphism groups, and the complexity of their isomorphism*, Beijing Journal of Pure and Applied Mathematics, in press. Available at arxiv.org/pdf/2410.02248.pdf
- [3] M. Rubin, *On the reconstruction of $\aleph_0$ -categorical structures from their automorphism groups*, Proceedings of the London Mathematical Society 3(2) (1994): 225–249.

3. TAKAYUKI KIHARA

The following problems concern a realizability-theoretic refinement of many-one and Wadge reducibility.

For sets $A, B \subseteq \omega^\omega$, define

$$A \leq_m B$$

if there is a computable function

$$\theta: \omega^\omega \rightarrow \omega^\omega$$

such that

$$x \in A \iff \theta(x) \in B$$

for every $x \in \omega^\omega$. Replacing “computable” by “continuous” gives Wadge reducibility, denoted by

$$A \leq_W B.$$

For formulas, the truth-preserving map on instances is not sufficient. The witnesses for the two formulas must also be uniformly convertible.

Let $\alpha \Vdash \varphi(x)$ mean that α is a witness, or realizer, for the truth of $\varphi(x)$. The witness relation is defined inductively from the logical structure of φ ; see [2, Definition 2.10].

A formula φ is realizability-theoretically many-one reducible to a formula ψ , written

$$\varphi \leq_m \psi,$$

if there are computable functions η, r^-, r^+ such that

(1)

$$\varphi(x) \iff \psi(\eta(x));$$

(2) If $\alpha \Vdash \varphi(x)$, then

$$r^-(\alpha, x) \Vdash \psi(\eta(x));$$

(3) If $\beta \Vdash \psi(\eta(x))$, then

$$r^+(\beta, x) \Vdash \varphi(x).$$

This is also called *Levin reducibility*; see [2, Definition 2.13]. Its continuous analogue gives the corresponding realizability-theoretic version of Wadge reducibility.

Existentially quantified variables may depend on all preceding universally quantified variables. For example, a witness for

$$\forall a \exists b \forall c \exists d \forall e \exists f \forall g \varphi(a, b, c, d, e, f, g, x)$$

consists of three functions

$$F_0: \omega \rightarrow \omega, \quad F_1: \omega^2 \rightarrow \omega, \quad F_2: \omega^3 \rightarrow \omega$$

such that

$$\forall a \forall c \forall e \forall g \varphi(a, F_0(a), c, F_1(a, c), e, F_2(a, c, e), g, x).$$

Thus, $F_0(a)$ witnesses b , $F_1(a, c)$ witnesses d , and $F_2(a, c, e)$ witnesses f .

In addition to the ordinary quantifiers $\exists$ and $\forall$, define

$$\exists^\infty n \varphi(n) \iff \forall m \exists n \geq m \varphi(n),$$

and

$$\forall^\infty n \varphi(n) \iff \exists m \forall n \geq m \varphi(n).$$

A finite sequence

$$\overline{Q} \in \{\exists, \forall, \exists^\infty, \forall^\infty\}^{<\omega}$$

is called a *quantifier pattern*. If

$$\overline{Q} = Q_0 Q_1 \cdots Q_\ell,$$

then a formula of the form

$$Q_0 n_0 Q_1 n_1 \cdots Q_\ell n_\ell \theta(n_0, \dots, n_\ell, x),$$

where θ is bounded, is called an $\overline{Q}$ -formula.

For each quantifier pattern $\overline{Q}$, let

$$\langle \overline{Q} \rangle$$

denote a fixed $\overline{Q}$ -complete formula.

Consider

$$\text{Bdd} := \{x \in \omega^\omega : \exists n \forall k x(k) \leq n\},$$

and

$$\text{FIN} := \{x \in \omega^\omega : \exists n \forall m \geq n x(m) = 0\}.$$

Although both are classically Σ_2^0 properties, their witness structures are different. The first has the quantifier pattern $\exists\forall$, whereas the second has the pattern $\forall^\infty$.

The realizability-theoretic classification of Σ_2^0 quantifier patterns consists of exactly three degrees:

$$\langle \forall^\infty \rangle <_m \langle \forall^\infty \forall \rangle <_m \langle \exists \forall \rangle.$$

The abstract treatment of this classification is given in [1].

The more concrete analysis of the Σ_3^0 and Π_3^0 levels is given in [2]. The known numbers of equivalence classes are as follows:

- (1) There is exactly one many-one equivalence class of Π_2^0 quantifier patterns, represented by

$$\forall\exists.$$

- (2) There are exactly three many-one equivalence classes of Σ_2^0 quantifier patterns, represented by

$$\exists\forall, \quad \forall^\infty\forall, \quad \forall^\infty.$$

- (3) There are exactly three many-one equivalence classes of Σ_3^0 quantifier patterns, represented by

$$\exists\forall\exists, \quad \forall^\infty\exists^\infty, \quad \forall^\infty\exists.$$

- (4) There are exactly five many-one equivalence classes of Π_3^0 quantifier patterns, represented by

$$\forall\exists\forall, \quad \exists^\infty\forall^\infty\forall, \quad \exists^\infty\forall, \quad \forall\forall^\infty\forall, \quad \forall\forall^\infty.$$

The corresponding classifications at the fourth level are still in progress.

Problem 3.1. Determine the equivalence classes of quantifier patterns at the fourth level of the arithmetical hierarchy:

- (1) How many $\equiv_m$ -equivalence classes of Σ_4^0 quantifier patterns are there?
- (2) How many $\equiv_m$ -equivalence classes of Π_4^0 quantifier patterns are there?
- (3) How many $\equiv_{dm}$ -equivalence classes of Σ_4^0 quantifier patterns are there?

Problem 3.2. Is there a decision procedure which, given quantifier patterns

$$\overline{P}, \overline{Q} \in \{\exists, \forall, \exists^\infty, \forall^\infty\}^{<\omega},$$

decides whether

$$\langle \overline{P} \rangle \leq_m \langle \overline{Q} \rangle?$$

References

- [1] T. Kihara, *Many-one reducibility with realizability*. arXiv:2403.16027 (2024).
- [2] T. Kihara, *The arithmetical hierarchy: a realizability-theoretic perspective*. arXiv:2410.15795 (2024).

4. LIANG YU

A set $A \subseteq \mathbb{R}^2$ is called a *Kekeya set* if, for every $\alpha \in \mathbb{R}$, there exists $C_\alpha \in \mathbb{R}$ such that

$$y = \alpha x + C_\alpha \implies (x, y) \in A$$

for every $(x, y) \in \mathbb{R}^2$. Equivalently,

$$\{(x, y) \in \mathbb{R}^2 : y = \alpha x + C_\alpha\} \subseteq A$$

for every $\alpha \in \mathbb{R}$.

Problem 4.1. Is there a Kekeya set $A \subseteq \mathbb{R}^2$ such that

$$x \oplus y \geq_T \emptyset'$$

for every $(x, y) \in A$?

A Kekeya set A is called *minimal* if every proper subset $B \subsetneq A$ is not a Kekeya set.

Problem 4.2. Does every Borel Kekeya set A contain a minimal Kekeya set $B \subseteq A$?

Yinghe Peng has shown that there is a full fine Kekeya set which contains no minimal Kekeya set.

5. JIALIANG HE

These notes concern two related definability questions in infinite combinatorics and Borel model theory.

Definition 5.1. A set $X \subseteq \mathbb{R}^2$ is called a *two-point set*, if for every line $L \subseteq \mathbb{R}^2$,

$$|X \cap L| = 2.$$

Theorem 5.2 (Larman [1]). There is no F_σ two-point set.

Since every F_σ set is Borel, Larman's theorem rules out the lowest nontrivial possibility in the Borel hierarchy, but it does not settle the following problem.

Problem 5.1. Is there a Borel two-point set?

Miller showed that, assuming Gödel's axiom of constructibility $V = L$, there is a Π_1^1 , i.e., coanalytic two-point set [2].

Definition 5.3. A *standard Borel space* is a measurable space isomorphic to the Borel space of a Polish space. A *Borel field* is a field $(F, +, \times)$ whose universe is a standard Borel space and whose field operations $+$, $\times$ are Borel.

Definition 5.4. Let F be a Borel field. A *Borel algebraically closed extension* of F is a Borel field K , together with a Borel injective field homomorphism

$$\iota : F \hookrightarrow K,$$

such that K is algebraically closed.

Problem 5.2 (Montalbán–Nies [3, Question 2.8]). Does every Borel field have a Borel algebraically closed extension?

References

- [1] D. G. Larman, *A problem of incidence*, Journal of the London Mathematical Society **43** (1968), 407–409.
- [2] A. W. Miller, *Infinite combinatorics and definability*, Annals of Pure and Applied Logic **41** (1989), no. 2, 179–203.
- [3] A. Montalbán and A. Nies, *Borel structures: a brief survey*, in *Effective Mathematics of the Uncountable*, Lecture Notes in Logic, vol. 41, Association for Symbolic Logic, La Jolla, CA, 2013, 124–134.

6. CHIEU-MINH TRAN

It is about Morley’s categoricity in power theorem. According to [1], there is the following consequence.

Theorem 6.1. For a countable theory T , if it is κ categorical for some $\kappa > \aleph_0$, then it is λ categorical for all uncountable λ .

And [2] replaced κ in the above theorem by recursion theoretic complexity (arithmetic degree). Then, the sequel goes to categoricity without Choice. The first research about this topic date back to [3] shows that we still have the theorem holds if we assume $\text{ZF} +$ there is some uncountable well ordered set of reals.

Theorem 6.2. $[\text{ZF} + \text{there is an uncountable well ordered set of reals}]$.

The following conditions on a countable (first order) T are equivalent:

- (1) T is categorical in some cardinal $\aleph_\alpha > \aleph_0$, in $\mathbf{V}$, of course
- (2) T is categorical in every cardinal $\aleph_\beta > \aleph_0$, in $\mathbf{V}$, of course

- (3) T is, in $\mathbf{L}[T]$, totally transcendental (i.e. $\aleph_0$ -stable) with no two cardinal models (i.e., for no model M of T and formula $\varphi(x, y) \in \mathbf{L}(\tau_T)$ and $\bar{a} \in {}^{\ell g(\bar{y})}M$ do we have $\aleph_0 \leq |\varphi(M, \bar{a})| < \|M\|$ and $\|M\|$ is a cardinal, i.e. the set of elements of M is well-orderable hence its power is a cardinal)
- (4) If $\mathbf{V}' \subseteq \mathbf{V}$ is a transitive class extending $\mathbf{L}$, $T \in \mathbf{V}'$ and $\mathbf{V}'$ satisfies ZFC, then the conditions in (3) hold
- (5) For some $\mathbf{V}'$ clause (4) holds.

However, for non-well-ordered cardinals, the situation deteriorates quickly. The sequel is the following. For such a theory T , we want to show it is very close to a strongly minimal T' . It is like a vector space over fixed field or algebraically closed field. And since there is no well order, it is hard to take a basis and match them to one another. So there is the question.

Problem 6.1. For vector space over fixed $\mathbb{K}$, is choose a basis require the full Axiom of Choice?

References

- [1] M. Morley, *Categoricity in power*, Transactions of the American Mathematical Society 114 (1965), 514-538.
- [2] J.L Goh, C.-M. Tran, *Categoricity without power*, 2026, arXiv:2605.00697
- [3] S. Shelah, *Model theory without choice: Categoricity*, The Journal of Symbolic Logic 74 (2) (2009): 361-401.

7. THEODORE SLAMAN

In 1954, [1] shows that for $E \subseteq \mathbb{R}^2$ an analytic set with Hausdorff dimension 1. Then, for almost all θ , the projection of E onto a line with slope θ has Hausdorff dimension 1. Later in 1969, [2] shows that CH could imply there is some $E \subseteq \mathbb{R}^2$ with Hausdorff dimension 1 and for all θ , projection of E onto a line of slope θ has Hausdorff dimension 0.

Problem 7.1. Is there a Davies example provably in ZFC?

References

- [1] J. M. Marstrand, *The dimension of Cartesian product sets*, Mathematical Proceedings of the Cambridge Philosophical Society 50 (1954), no. 2, 198-202.
- [2] R. O. Davies, *Measures of Hausdorff type*, Journal of the London Mathematical Society s2-1 (1969), no. 1, 30-34.

8. HANG ZHANG

The following questions are motivated by the problem whether hyper-hyperfiniteness implies hyperfiniteness.

Let M be a countable model for a sufficiently large fragment of ZFC. Suppose that a partial order P is in M . Define the space Gen_M^P of M -generics for P by

$$Gen_M^P = \{G \subseteq P : G \text{ is an } P\text{-generic filter over } M\}.$$

If we identify Gen_M^P as a subspace of 2^P with the product topology, then Gen_M^P is G_δ in 2^P and thus a Polish space. See [1] for details. Define $G \sim H$ if and only if $M[G] = M[H]$. Consider the equivalence relation $\sim$ on Gen_M^P .

Example 8.1. (Calderoni–Sinapova [2]) Let M be a countable model for large fragment ZFC. Suppose $M \models \kappa$ is measurable (witnessed by $\mathcal{U}$). If $M \models \mathbb{P} = \{(s, A) : s = \langle s_0, \dots, s_n \rangle, n \in \omega, A \in \mathcal{U}\}$, with order $(s, A) \leq (t, B) \iff t \subseteq s \wedge s \setminus t \subseteq B \wedge A \subseteq B$. Then for any generic sequences $\vec{\alpha}, \vec{\beta}$ of $\mathbb{P}$ over M ,

$$(*) \quad M[\vec{\alpha}] = M[\vec{\beta}] \iff \exists N \forall n > N \quad \alpha(n) = \beta(n).$$

Problem 8.1. Find more models M and forcing notions $\mathbb{P} \in M$ such that $(*)$ is true.

Remark 8.2 ([1, Theorem 3.1]). If $\mathbb{P} = \mathbb{C}$ (Cohen forcing), then " $x \sim y \iff M[x] = M[y]$ " is hyper-hyperfinite.

Problem 8.2 ([1, Question 5.2]). Is $\sim$ hyperfinite?

Problem 8.3. Suppose there are a model M and a forcing notion $\mathbb{P} \in M$ such that $(*)$ is true. Can we deduce that $M \models [\exists \text{ a large cardinal}]$?

Remark 8.3. If

$$\langle 0, \dots, 0, 1, 0, \dots, 0, 1, 1, 0, \dots \rangle$$

is Cohen, then

$$\langle 1, \dots, 1, 0, 1, \dots, 1, 0, 0, 1, \dots \rangle$$

is also Cohen. Notice that these two Cohen reals lead to the same forcing extension. So Cohen forcing does not satisfy $(*)$ for any M .

References

- [1] I. B. Smythe, *Equivalence of generics*, Archive for Mathematical Logic 61(2022), 795–812.

- [2] F. Calderoni, D. Sinapova, *Forcing, genericity and countable Borel equivalence relations*, preprint, 2025, arXiv: 2503.14811.

9. DANIEL TURETSKY

If $\mathbf{a} < \mathbf{b} < \mathbf{c}$ are Turing degrees, and $\mathbf{a}$, $\mathbf{c}$ contain different Martin-Löf randoms, then must $\mathbf{b}$ contain one? The answer is no.

Problem 9.1. Does this hold with a larger degree notion, for example, LR-degrees?

10. PHILIP WELCH

First we recall some basic notions in [1]. Let β be an ordinal. Suppose M is a nonstandard model of KP, and $\{\zeta_i : i \in \mathbb{N}\}$, $\{s_i : i \in \mathbb{N}\}$ are sequences of M -ordinals. If $(M, \{\zeta_i : i \in \mathbb{N}\}, \{s_i : i \in \mathbb{N}\})$ satisfies the following:

- (1) M extends L_β ;
- (2) for all i , $\zeta_i < \zeta_{i+1} < \beta$ and $s_{i+1} <^M s_i$;
- (3) for all i , $L_{\zeta_i} \prec_{\Sigma_{m+1}} L_{s_i}^M$;
- (4) for all i , $L_{s_{i+1}}^M \prec_{\Sigma_{m-1}} L_{s_i}^M$,

then we call this triple a strong Σ_{m+1} -nesting on β . And we say β supports a strong Σ_{m+1} -nesting if there exist such nonstandard model M of KP and sequences $\{\zeta_i : i \in \mathbb{N}\}, \{s_i : i \in \mathbb{N}\}$ of M -ordinals for L_β .

We call a function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ an operator ($\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ are identified). An operator Φ is said to be

- (1) Π_m^1 if there exists a Π_m^1 formula ϕ and a parameter Y such that

$$\forall n \forall X (n \in \Phi(X) \leftrightarrow \phi(n, X, Y));$$

- (2) monotone if $X \subseteq Y$ implies $\Phi(X) \subseteq \Phi(Y)$.

The axiom $\Pi_{m+1}^1\text{-MI}$ asserts that for every monotone Π_{m+1}^1 operator Φ , there is some prewellordering W such that for all $a \in \text{field}(W) = \text{dom}(W) \cup \text{ran}(W)$, we have

$$W_a = \Phi(W_{<a}).$$

Here

$$\begin{aligned} W_a &= \{b : a <_W b\}, \\ W_{<a} &= \bigcup \{W_b : b <_W a\}. \end{aligned}$$

Note that an operator Φ can be iterated by setting $\Phi^0 = \emptyset$, $\Phi^{<\alpha} = \bigcup_{\gamma < \alpha} \Phi^\gamma$ and $\Phi^\alpha = \Phi(\Phi^{<\alpha})$. Put $\Phi^\infty = \bigcup_\alpha \Phi^\alpha$. We can see that the field of W is Φ^∞ .

Problem 10.1. If $L_\beta \models V = HC$, then

$$\left(\mathcal{P}(\omega) \cap L_\beta \models \Pi_{m+1}^1\text{-MI} \right) \iff \beta \text{ supports a strong } \Sigma_{m+1}\text{-nesting.}$$

References

- [1] J. P. Aguilera, P. D. Welch, *Determinacy on the edge of second-order arithmetic, I*, Journal of the London Mathematical Society 113(3) (2026), e70456.

11. SU GAO

This question is about equivalence relations given by complete first-order theories. The question has been circulating in the community since early 2000s.

Consider a complete first order theory T in a countable language. Let $\text{Mod}(T)$ be the Polish space of countable models of T with universe ω . Let $\cong|_{\text{Mod}(T)}$ denote the isomorphism relation on $\text{Mod}(T)$. Recall two basic equivalence relations in descriptive set theory:

- (1) $=$ is the identity relation on $\mathbb{R}$;
- (2) $=^+$ is the equivalence relation on $\mathbb{R}^\omega$ that $(x_n)_{n \in \omega} =^+ (y_n)_{n \in \omega}$ if they represent the same countable subset of reals, i.e., if
$$\{x_n : n \in \omega\} = \{y_n : n \in \omega\}.$$

Also note the following fact:

Fact 11.1. (1) There is a complete theory T such that $=$ is Borel bi-reducible to $\cong|_{\text{Mod}(T)}$. For example, T can be chosen as the theory of the equivalence relation on ω with infinitely many equivalence classes of any finite size.
(2) There is a complete theory T' such that $=^+$ is Borel bi-reducible to $\cong|_{\text{Mod}(T')}$. For example, T' can be chosen as the theory of infinitely many independent equivalence relations on ω and every equivalence relation has infinitely many equivalence classes of any finite size.

So we have the following natural question:

Problem 11.1. Does there exist a complete first-order theory T in a countable language such that $= <_B \cong|_{\text{Mod}(T)} <_B =^+$?

Remark 11.2. Note that $= <_B =^+$, and there are equivalence relations strictly between them. For example, $E_0, E_\infty, E_0^\omega$. So, they are natural candidates for the complexity of $\cong|_{\text{Mod}(T)}$ for some T if the problem has a positive answer.

There have been some research (e.g. [1, 2, 3, 4]) on this problem and more generally on the isomorphism relation $\cong|_{\text{Mod}(T)}$ in the Borel reducibility sense. In particular, Dave Marker showed in [1] that if $S(T)$ (the Stone space of T -types) is uncountable, then T is not an example for Problem 11.1.

References

- [1] D. Marker, *The Borel complexity of isomorphism for theories with many types*, Notre Dame Journal of Formal Logic 48 (2007), no. 1, 93–97.
- [2] M. Koerwien, *Comparing Borel reducibility and depth of an ω -stable theory*, Notre Dame Journal of Formal Logic 50 (2009), no. 4, 365–380.
- [3] S. Coskey, R. Kossak, *The complexity of classification problems for models of arithmetic*, Bulletin of Symbolic Logic 16 (2010), no. 3, 345–358.
- [4] R. Rast, D. Singh Sahota, *The Borel complexity of isomorphism for o-minimal theories*, The Journal of Symbolic Logic 82 (2017), no. 2, 453–473.